UNIVERZITA KARLOVA
Matematicko-fyzikální
fakulta

Dr. Malte Kampschulte
Dr. Alexei Gazca

Advanced Methods in Mathematical Analysis

Winter Semester 2025/26 —— Sheet 10

Task 1

Take $\Omega = (0,1)$ with the Lebesgue measure and consider the characteristic function $f(t) = \chi_{(0,t)}$ for $t \in \Omega$.

(a) Prove that f is not strongly measurable considered as a function $f\colon \Omega \to L^\infty(0,1)$.

(b) Prove that f is strongly measurable considered as a function $f\colon \Omega \to L^2(0,1)$.

Hint. Recall that a function $f\colon \Omega \to X$ into a Banach space X is strongly measurable if and only if it is weakly measurable ($t \in \Omega \mapsto \langle v^*, f(t)\rangle$ is measurable for all $x^* \in X^*$) and it is almost separably valued (there is a null set $E \subset \Omega$ such that $f(\Omega \setminus E)$ is separable).

Task 2

Prove that a strongly measurable function $f\colon \Omega \to X$ is Bochner integrable if and only if

$$\int_\Omega \|f\| \,\mathrm{d}\mu < +\infty,$$

and in this case

$$\left\| \int_\Omega f \,\mathrm{d}\mu \right\| \le \int_\Omega \|f\| \,\mathrm{d}\mu.$$

Task 3 (Vector Dominated Convergence Theorem)

Take a measurable space (Ω, Σ, μ) and a Banach space X. Let $f\colon \Omega \to X$ be strongly measurable and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Bochner integrable functions satisfying $\|f_n(\omega) - f(\omega)\| \to 0$ for μ-a.e. $\omega \in \Omega$. Suppose there is a non-negative (Lebesgue) integrable function $g\colon \Omega \to \mathbb{R}$ such that $\|f_n\| \le g$ μ-a.e. for all $n \in \mathbb{N}$. Prove that f is Bochner integrable and for each $E \in \Sigma$ one has

$$\lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu.$$

Definition. Let $[a,b]$ be a compact interval with positive length. A *tagged partition* $\dot{\mathcal{P}} = (\mathcal{P}, \{x_i^*\}_{i=1}^n)$ consists of a partition $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$ of $[a,b]$ and a set of *tags* $\{x_i^*\}_{i=1}^n$ such that $x_i^* \in [x_{i-1}, x_i]$ for all $i \in \{1, \dots, n\}$. The norm of a tagged partition is defined as $\|\dot{\mathcal{P}}\| := \max_{i\in\{1,\dots,n\}}(x_i - x_{i-1})$

The *Riemann sum* of a bounded function $f\colon [a,b] \to X$ into a Banach space X with respect to the tagged partition $\dot{\mathcal{P}}$ is defined as

$$\mathcal{R}(f, \dot{\mathcal{P}}) := \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}).$$

The function $f\colon [a,b] \to X$ is said to be *Riemann integrable* with integral $\Lambda \in X$ if

$$\Lambda = \lim_{\|\dot{\mathcal{P}}\|\to 0} \mathcal{R}(f, \dot{\mathcal{P}}),$$

meaning that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|\mathcal{R}(f, \dot{\mathcal{P}}) - \Lambda\| \le \varepsilon,$$

for all tagged partitions $\dot{\mathcal{P}}$ with $\|\dot{\mathcal{P}}\| \le \delta$. In this case one writes $\Lambda = \int_a^b f(x)\,\mathrm{d}x$.

Task 4 (Riemann vs. Bochner integral)

Let X be a Banach space and $f\colon [0,1] \to X$ continuous. Prove that f is Bochner integrable and that its Bochner and Riemann integrals coincide.