Numerical Analysis for Nonlinear PDE

Summer Semester 2026 —— Sheet 9

Task 1 (Compactness of DG approximations) (4+6+6 Points)

For given polynomial degrees $k, l \in \mathbb{N}$, define the lifting operator $\mathcal{R}_h^l \colon \mathbb{P}^k(\mathcal{T}_h) \to \mathbb{P}^l(\mathcal{T}_h)$ through

$$\int_\Omega \mathcal{R}_h^l(v_h) \cdot \boldsymbol{\varphi}_h = \sum_{F \in \mathcal{F}_h^i} \int_F [\![v_h \boldsymbol{n}]\!]_F \cdot \{\!\{ \boldsymbol{\varphi}_h \}\!\} \qquad \forall\, \boldsymbol{\varphi}_h \in \mathbb{P}^l(\mathcal{T}_h)^d.$$

Define the DG ('Discontinuous Galerkin') gradient operator $\mathcal{G}_h^l \colon \mathbb{P}^k(\mathcal{T}_h) \to \mathbb{P}^{\max\{k-1,l\}}(\mathcal{T}_h)^d$ as $\mathcal{G}_h^l(v_h) := \nabla_h v_h - \mathcal{R}_h^l(v_h)$.

(a) Prove that for any $p \in [1, \infty)$ and $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ one has

$$\|\mathcal{R}_h^l(v_h)\|_{L^p(\Omega)}^p \lesssim \sum_{F \in \mathcal{F}_h^i} \int_F \frac{1}{h_F^{p-1}} |[\![v_h]\!]_F|^p.$$

(b) Let $\{u_h\}_{h>0}$ be a sequence with $u_h \in \mathbb{P}^k(\mathcal{T}_h)$ such that

$$\|\nabla_h u_h\|_{L^p(\Omega)}^p + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![u_h]\!]_F|^p \le c,$$

for some c indepenedent of h and $p \in (1, \infty)$. Prove that, up to a subsequence, we have as $h \to 0$:

$$\begin{aligned} u_h &\to u && \text{strongly in } L^p(\Omega), \\ \mathcal{G}_h^l(u_h) &\rightharpoonup \nabla u && \text{weakly in } L^p(\Omega)^d, \end{aligned}$$

for some $u \in W_0^{1,p}(\Omega)$.

(c) Define the discrete energy $I_h \colon \mathbb{P}^k(\mathcal{T}_h) \to \mathbb{R}$ for $p \in (1, \infty)$ as

$$I_h(v_h) := \frac{1}{p} \int_\Omega |\mathcal{G}_h^l(v_h)|^p + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![v_h]\!]_F|^p - \int_\Omega f v_h$$

where $f \in L^{p'}(\Omega)$ is given. Prove that there is a unique minimiser $u_h \in \mathbb{P}^k(\mathcal{T}_h)$, and that u_h converges (as $h \to 0$) to the solution of the p-Laplace problem $u \in W_0^{1,p}(\Omega)$.