Numerical Analysis for Nonlinear PDE

Summer Semester 2026 —— Sheet 8

Task 1 (Commuting properties) (4 Points)

Prove that the Crouzeix–Raviart interpolation operator $\mathcal{I}_h^{\mathrm{CR}} \colon H^1(\Omega) \to V^h$ satisfies the following commuting properties:

$$\nabla_h(\mathcal{I}_h^{\mathrm{CR}}(v)) = \Pi_h(\nabla v) \qquad \forall\, v \in H^1(\Omega),$$
$$\mathrm{div}_h(\mathcal{I}_h^{\mathrm{CR}}(\boldsymbol{v})) = \Pi_h \operatorname{div} \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Task 2 (Applications of quasi-interpolation operators) (3+3 Points)

With the help of the nodal averaging operator $\mathcal{I}_h^{\mathrm{av}} \colon V_D^h \to \mathbb{P}^1(\mathcal{T}_h) \cap H_0^1(\Omega)$ from the lecture, prove the following:

(a) Any $v_h \in V_D^h$ satisfies the discrete Poincaré inequality:

$$\|v_h\|_\Omega \le c_P \|\nabla_h v_h\|_\Omega.$$

(b) If $\{v_h\}_{h>0}$ is a sequence with $v_h \in V_D^h$ that is uniformly bounded ($\|\nabla_h v_h\|_\Omega < c$), then, up to subsequences, we have as $h \to 0$:

$$v_h \to v \qquad \text{strongly in } L^2(\Omega),$$
$$\nabla_h v_h \rightharpoonup \nabla v \qquad \text{weakly in } L^2(\Omega)^d,$$

for some $v \in H_0^1(\Omega)$.

Definition. The local (lowest-order) Raviart–Thomas element on a mesh element $T \in \mathcal{T}_h$ is defined as $\mathrm{RT}_0(T) := \mathbb{P}^0(T)^d \oplus \boldsymbol{x}\,\mathbb{P}^0(T) = \{\boldsymbol{a} + b\,\boldsymbol{x} \mid \boldsymbol{a} \in \mathbb{R}^d,\, b \in \mathbb{R}\}$. We define the global Raviart–Thomas finite element space as:

$$\Sigma^h := \mathrm{RT}_0(\mathcal{T}_h) := \{\boldsymbol{r}_h \in H(\mathrm{div};\Omega) \mid \boldsymbol{r}_h|_T \in \mathrm{RT}_0(T)\ \forall\, T \in \mathcal{T}_h\}.$$

Task 3 (Crouzeix–Raviart and Raviart–Thomas elements) (3+3 Points)

(a) Prove that integration by parts is possible with Crouzeix–Raviart and Raviart–Thomas elements:

$$(\boldsymbol{r}_h, \nabla_h v)_\Omega + (\operatorname{div} \boldsymbol{r}_h, v_h)_\Omega = (\boldsymbol{r}_h \cdot \boldsymbol{n}, v_h)_{\partial\Omega} \qquad \forall\, v_h \in V^h,\ \boldsymbol{r}_h \in \Sigma^h.$$

Hint. Recall that $H(\mathrm{div};\Omega)$ functions have continuous normal jumps across facets, and notice that $\boldsymbol{r}_h|_F \cdot \boldsymbol{n} \in \mathbb{P}^0(F)$ on any mesh facet F.

(b) Suppose that $f \in \mathbb{P}^0(\mathcal{T}_h)$, and that $u_h \in V_D^h$ satisfies $(\nabla_h u_h, \nabla_h v_h)_\Omega = (f, v_h)_\Omega$ for all $v_h \in V_D^h$ (i.e. u_h is a discrete solution of Laplace). Define the flux

$$\boldsymbol{q}_h := \nabla_h u_h - \frac{f}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}) \in \mathbb{P}^1(\mathcal{T}_h),$$

where $\mathrm{id}_{\mathbb{R}^d}$ represents the identity on $\mathbb{R}^d$. Prove that $\boldsymbol{q}_h \in \Sigma^h$, and $\operatorname{div} \boldsymbol{q}_h = -f$.

Hint. Evaluate $\int_F [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!]_F \phi_F$ for all $F \in \mathcal{F}_h^i$ ($\phi_F \in V_D^h$ is a basis function).