Numerical Analysis for Nonlinear PDE

Summer Semester 2026 — Sheet 7

Task 1 (Constrained minimisation) (2+2+2 Points)

(a) Let K be a closed, convex, non-empty subset of a normed space V. Find $\partial\chi_K(v)$ for any $v \in K$.

(b) Let $J\colon V \to \mathbb{R}$ be a Gâteaux-differentiable function defined on a Banach space V. Prove that u is a solution of the constrained minimisation problem $J(u) = \min_{v\in K} J(v)$, with $K \subset V$ a closed, convex, non-empty subset, if and only if

$$\langle J'(u), v - u\rangle_V \geq 0 \qquad \text{for all } v \in K.$$

(c) Take the usual Laplace energy $I(v) := \frac{1}{2}\|\nabla v\|_\Omega^2 - \langle f, v\rangle_\Omega$, defined for $v \in H^1(\Omega)$ with $f \in H^{-1}(\Omega)$ given. Derive the optimality condition for a minimiser $u \in H^1(\Omega)$ with the constraint $u|_{\partial\Omega} = u_b$, where $u_b \in H^{1/2}(\partial\Omega)$ is a given boundary condition. Can you turn the optimality condition into an *equation*?

Task 2 (Computing dual problems: p-Laplace) (3+3 Points)

(a) Let $(V, \|\cdot\|)$ be a normed space and $(V^*, \|\cdot\|_*)$ its dual. Let $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a convex lsc function that is even ($\varphi(t) = \varphi(-t)$). Prove that the function

$$f\colon V \to \mathbb{R} \cup \{+\infty\}$$
$$f(v) = \varphi(\|v\|)$$

is a proper convex lsc function, and

$$f^*(v^*) = \varphi^*(\|v^*\|_*)$$

Hint. Write $f^*(v^*) = \sup_{t\geq 0} \sup_{v\in V,\, \|v\|=t} [\langle v^*, v\rangle - \varphi(\|v\|)]$.

(b) Use (a) and the Fenchel duality theorem (Prop. 6.2.1) to find a dual problem for the p-Laplace problem ($p \in (1, \infty)$), which consists in the minimisation of the primal energy:

$$I\colon W_0^{1,p}(\Omega) \to \mathbb{R}, \tag{1}$$
$$I(v) := \frac{1}{p}\int_\Omega |\nabla v|^p - (f, v)_\Omega, \tag{2}$$

with $f \in L^{p'}(\Omega)$ given.

Task 3 (Computing dual problems: Bingham) (4 Points)

Using the Fenchel duality theorem, find a dual problem for the scalar Bingham problem, which consists in the minimisation of the energy $I\colon H^1_0(\Omega) \to \mathbb{R}$, defined as:

$$I(v) := \frac{\nu_\star}{2} \int_\Omega |\nabla v|^2 + \tau_\star \int_\Omega |\nabla v| - \int_\Omega f v, \tag{3}$$

with $\nu_\star > 0$, $\tau_\star \geq 0$, and $f \in L^2(\Omega)$ given.

Hint. You may use the fact that if $g\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function such that $\boldsymbol{r} \in L^p(\Omega)^d \mapsto g(\cdot, \boldsymbol{r}(\cdot)) \in L^q(\Omega)$ is continuous ($p, q \in [1, \infty)$), then

$$\sup_{\boldsymbol{r} \in L^p(\Omega)^d} \int_\Omega g(x, \boldsymbol{r}(x))\,\mathrm{d}x = \int_\Omega \sup_{\boldsymbol{a} \in \mathbb{R}^d} g(x, \boldsymbol{a})\,\mathrm{d}x.$$

At some point, the characterisation of *equality* in the Fenchel–Young inequality could be useful.