Numerical Analysis for Nonlinear PDE

Summer Semester 2026 —— Sheet 6

Task 1 (The Prager–Synge identity) (2+3 Points)

Take the primal and dual energy functionals corresponding to the Laplace problem:

$$I\colon H^1_0(\Omega) \to \mathbb{R} \cup \{+\infty\}, \qquad D\colon L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\},$$
$$I(v) = \tfrac{1}{2}\|\nabla v\|^2_\Omega - (f, v)_\Omega, \qquad D(v) = -\tfrac{1}{2}\|\boldsymbol{r}\|^2_\Omega - \chi^\Omega_{\{-f\}}(\operatorname{div} \boldsymbol{r}),$$

with $f \in L^2(\Omega)$. Denote by $u \in H^1_0(\Omega)$ and $\boldsymbol{q} := \nabla u \in L^2(\Omega)^d$ the primal and dual solution, respectively.

(a) Prove that the strong duality relation $I(u) = D(\boldsymbol{q})$ holds.

(b) Prove that for any $\boldsymbol{r} \in L^2(\Omega)^d$ with $\operatorname{div} \boldsymbol{r} = -f$ and $v \in H^1_0(\Omega)$ one has:

$$D(\boldsymbol{r}) - D(\boldsymbol{r}) = \tfrac{1}{2}\|\boldsymbol{r} - \boldsymbol{q}\|^2_\Omega \qquad I(v) - D(\boldsymbol{r}) = \tfrac{1}{2}\|\nabla v - \boldsymbol{r}\|^2_\Omega.$$

Conclude that the Prager–Synge identity holds:

$$\|\nabla v - \nabla u\|^2_\Omega + \|\boldsymbol{r} - \boldsymbol{q}\|^2_\Omega = \tfrac{1}{2}\|\nabla v - \boldsymbol{r}\|^2_\Omega.$$

Task 2 (Computing convex conjugates) (2+2+2 Points)

For the following functions compute f^* and f^{**}:

(a) $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = a^\top x + b$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

(b) $f\colon \mathbb{R}^n \to \mathbb{R}$, $f(x) = x^\top A x$ with $A \in \mathbb{R}^{n\times n}$ symmetric and positive definite.

(c) $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.

Definition. The support function of a subset C of a normed vector space V is defined as

$$\sigma_C\colon V^* \to \mathbb{R} \cup \{+\infty\}$$
$$\sigma_C(v^*) := \sup_{v\in C}\langle v^*, v\rangle$$

Task 3 (Subdifferential of the norm) (2+3 Points)

Let C be a non-empty, closed, convex subset of a normed vector space V; denote the indicator function of C by χ_C.

(a) Prove that

$$(\chi_C)^* = \sigma_C \qquad \text{and} \qquad \sigma_C^* = \chi_C.$$

(b) Prove that the subdifferential of the norm is given by

$$\partial\|v\| = \{v^* \in V^* \mid \langle v^*, v\rangle = \|v\|,\ \|v^*\|_* \le 1\},$$

where $\|\cdot\|_*$ denotes the dual norm in V^*.