UNIVERZITA KARLOVA
Matematicko-fyzikální
fakulta

Dr. Alexei Gazca

Numerical Analysis for Nonlinear PDE

Summer Semester 2026 —— Sheet 5

Task 1 (Mixed formulations) (3+3+3+4+3 Points)

(a) Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Prove that there is a constant $\gamma > 0$ such that:

$$\inf_{v \in W_0^{1,p}(\Omega)} \sup_{\boldsymbol{r} \in L^{p'}(\Omega)} \frac{\int_\Omega \boldsymbol{r} \cdot \nabla v}{\|v\|_{W^{1,p}(\Omega)} \|\boldsymbol{r}\|_{L^{p'}(\Omega)}} \geq \gamma.$$

Hint: Poincaré's inequality.

(b) Let $\mathcal{B} \colon \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function that satisfies p'-growth and coercivity properties:

$$|\mathcal{B}(\boldsymbol{b})| \lesssim |\boldsymbol{b}|^{p'-1} \qquad |\boldsymbol{b}|^{p'} - 1 \lesssim \mathcal{B}(\boldsymbol{b}) \cdot \boldsymbol{b} \qquad \forall\, \boldsymbol{b} \in \mathbb{R}^d.$$

Derive a weak formulation for the system

$$\begin{cases} \mathcal{B}(\boldsymbol{q}) = \nabla u & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{q} = f & \text{in } \Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

with $f \in W^{-1,p'}(\Omega)$ given, for a weak solution $(\boldsymbol{q}, u)$ belonging to $L^{p'}(\Omega)^d \times W_0^{1,p}(\Omega)$.

(c) Take piecewise polynomial conforming approximations of degree $k \geq 1$ on shape-regular meshes $\{\mathcal{T}_h\}_{h>0}$: $V_h = \mathbb{P}^k(\mathcal{T}_h) \cap W_0^{1,p}(\Omega)$ for u and $\Sigma_h = \mathbb{P}^{k-1}(\mathcal{T}_h)^d$ for $\boldsymbol{q}$ (note that we do not impose continuity across elements in Σ^h). Prove the discrete analogue of the inf-sup inequality from (a):

$$\inf_{v_h \in V_h} \sup_{\boldsymbol{r}_h \in \Sigma_h} \frac{\int_\Omega \boldsymbol{r} \cdot \nabla v}{\|v\|_{W^{1,p}(\Omega)} \|\boldsymbol{r}\|_{L^{p'}(\Omega)}} \geq \gamma.$$

(d) Write down a discrete formulation on $\Sigma_h \times V_h$ and show that a discrete solution $(\boldsymbol{q}_h, u_h) \in \Sigma_h \times V_h$ exists.

Hint: If you don't know saddle point theory, you can consider the approximate discrete system with the additional term $\varepsilon u_{h,\varepsilon}$; this way the system is now coercive, so you can apply results from the lecture. Take the limit $\varepsilon \to 0$ to get existence of discrete solutions.

(e) Assuming that $\mathcal{B}$ is monotone, prove that (up to subsequences) $\boldsymbol{q}_h \rightharpoonup \boldsymbol{q}$ weakly in $L^{p'}(\Omega)$ and $u_h \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, as $h \to 0$, where $(\boldsymbol{q}, u)$ is a weak solution of the original system.

Hint: You'll need the inequality from (c) to get uniform bounds for u_h.

Task 2 (The incompressible Navier–Stokes equations) (2+3+4+3 Points)

In the steady Navier–Stokes system we look for a velocity field $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^d$ and a pressure $p\colon \Omega \to \mathbb{R}$ (again $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz and $d \in \{2,3\}$), such that

$$\begin{cases} -\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Here $\boldsymbol{f}\colon \Omega \to \mathbb{R}^d$ is a given forcing term.

(a) Assume that $\boldsymbol{f} \in H^{-1}(\Omega)^d$. Derive a weak formulation of the Navier–Stokes equations for a solution $(\boldsymbol{u}, p)$ in the space $H_0^1(\Omega)^d \times L^2(\Omega)$.

(b) Prove that for any $\boldsymbol{v} \in H_0^1(\Omega)^d$ with $\operatorname{div} \boldsymbol{v} = 0$ one has:

$$\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{v} = 0$$

(You should justify why this integral is well-defined in the first place.)

(c) Write down a discrete formulation with conforming FE spaces $V_h \times Q_h \subset H_0^1(\Omega)^d \times L^2(\Omega)$ (with shape-regular meshes and approximation properties, as always). Prove that a discrete velocity $\boldsymbol{u}_h \in V_h$ exists in the subspace

$$V_{h,\mathrm{div}} := \{ v_h \in V_h \mid (\operatorname{div} v_h, q_h)_\Omega = 0 \quad \forall\, q_h \in Q_h \}\,.$$

Hint: Examine the restriction of the discrete formulation to $V_{h,\mathrm{div}}$ and apply Proposition 2.1.2 or Corollary 2.1.1.

(d) Prove that $\boldsymbol{u}_h$ converges weakly in $H_0^1(\Omega)^d$ (up to a subsequence) to a function $\boldsymbol{u}$; what equation is satisfied by $\boldsymbol{u}$?

(e) [**Bonus +3 Points**] If you know about saddle point theory, prove that if the pair $V_h \times Q_h$ is inf-sup stable, then a discrete pressure $p_h \in Q_h$ also exists, and converges weakly to a continuous pressure $p \in L^2(\Omega)$.

Task 3 (Bingham model: alternative formulation) (4 Points)

In the lecture we studied the (scalar) Bingham system for $(\boldsymbol{q}, u) \in L^2(\Omega)^d \times H_0^1(\Omega)$:

$$\begin{cases} -\operatorname{div} \boldsymbol{q} = f & \text{in } \Omega, \\ \nu_\star \nabla u = \frac{(|\boldsymbol{q}| - \tau_\star)_+}{|\boldsymbol{q}|} \boldsymbol{q} & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nu_\star > 0$ is the viscosity, and $\tau_\star \geq 0$ the yield stress. Show that this system can be equivalently written as:

$$\begin{cases} -\operatorname{div} \boldsymbol{q} = f & \text{in } \Omega, \\ \boldsymbol{q} = \nu_\star \nabla u + \tau_\star \boldsymbol{\lambda} & \text{a.e. in } \Omega, \\ \boldsymbol{\lambda} \cdot \nabla u = |\nabla u|, \quad |\boldsymbol{\lambda}| \leq 1 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\boldsymbol{\lambda} \in L^2(\Omega)^d$ (this represents the plastic stress).