



## Numerical Analysis for Nonlinear PDE

Summer Semester 2026 — Sheet 4

### Task 1 (Existence of discrete solutions for monotone problems) (3 Points)

The goal of this problem is to be more precise in the application of Corollary 2.1.1 in the proof of existence of discrete solutions.

Suppose that  $A: V_n \rightarrow V_n^*$  is continuous and coercive on a finite dimensional space  $V_n$  (say  $\dim(V_n) = n$ ), and let  $f \in V_n^*$ . Prove that there exists  $u_n \in V_n$  such that  $A(u_n) = f$ .

**Hint:** Write functions  $w \in V_n$  in terms of an orthonormal basis  $w = \sum_{j=1}^n W_j \varphi_j$ , frame the problem in terms of the coefficients  $(W_j)_j \in \mathbb{R}^n$  and apply Corollary 2.1.1 on  $\mathbb{R}^n$ .

**[Bonus +4 Points]** Prove that if  $K_n$  is a non-empty, closed, convex subset of  $V_n$ , there is a unique  $u_n \in K_n$  such that

$$\langle A(u_n), v - u_n \rangle_{V_n} \geq \langle f, v - u_n \rangle_{V_n} \quad \forall v \in K. \quad (1)$$

**Hint:** Use the orthogonal projection onto  $K_n$

### Task 2 (The $p$ -Laplacian: variational approach) (3+3+2 Points)

Consider the energy functional  $J: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  for  $p \in (1, \infty)$ :

$$J(v) := \frac{1}{p} \int_{\Omega} |\nabla v|^p - \langle f, v \rangle_{W_0^{1,p}(\Omega)},$$

with  $f \in W^{-1,p'}(\Omega)$  given. Take conforming finite element spaces  $V_h \subset W_0^{1,p}(\Omega)$  associated to a family of shape-regular triangulations  $\{\mathcal{T}_h\}_{h>0}$ .

- Using the direct method of the calculus of variations, prove that there exists a unique minimiser  $u_h$  on  $V_h$ .
- Prove that the energy functionals

$$J_h(v_h) := \begin{cases} J(v_h) & \text{if } v_h \in V_h, \\ +\infty & \text{if } v_h \notin V_h. \end{cases}$$

$\Gamma$ -converge to  $J$  with respect to weak convergence in  $W_0^{1,p}(\Omega)$ .

- Conclude that  $u_h \rightharpoonup u$  weakly in  $W_0^{1,p}(\Omega)$ , where  $u$  is the unique minimiser of  $J$  on  $W_0^{1,p}(\Omega)$ .
- [Bonus]** Prove that  $u_h$  actually converges strongly to  $u$ .

### Task 3 (The $p$ -Laplacian: monotone operator theory approach) (2+2+2 Points)

Consider the same setting as in the previous problem.

- Prove that  $J$  is Fréchet-differentiable, and its derivative is given by

$$\langle J'(u), v \rangle_{W_0^{1,p}(\Omega)} = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v - \langle f, v \rangle_{W_0^{1,p}(\Omega)} \quad \forall u, v \in W_0^{1,p}(\Omega). \quad (2)$$



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- (b) Prove that the operator  $A: W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$  defined by  $\langle A(u), v \rangle_{W_0^{1,p}(\Omega)} := \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v$  is hemicontinuous, coercive and strictly convex.
- (c) Conclude that there is a unique  $u_h \in V_h$  that solves the discrete problem

$$\int_{\Omega} |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla v_h = \langle f, v_h \rangle_{W_0^{1,p}(\Omega)} \quad \forall v_h \in V_h, \quad (3)$$

and that  $u_h \rightharpoonup u$  weakly in  $W_0^{1,p}(\Omega)$ , where  $u$  is the unique solution to the  $p$ -Laplace equation:

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \langle f, v \rangle_{W_0^{1,p}(\Omega)} \quad \forall v \in W_0^{1,p}(\Omega). \quad (4)$$