Numerical Analysis for Nonlinear PDE

Summer Semester 2026 — Sheet 3

Task 1 (Time compactness for non-conforming methods) (6+5 Points)

Take a family of shape-regular triangulations $\{\mathcal{T}_h\}_{h>0}$ and associated finite element spaces $V_h \subset H^1(\Omega)$, as described in Section 2.6 of the lecture notes. Similarly take uniform partitions $\{\mathcal{J}_\tau\}_\tau$ of the interval $I := [0,T]$ into $m_\tau := \frac{T}{\tau}$ pieces of length τ as in Section 4.1. (So h is the mesh size and τ is the time step.) Consider the heat equation on $Q := I \times \Omega$:

$$\int_I \langle \partial_t u(t), v(t)\rangle_{H_0^1(\Omega)}\,\mathrm{d}t + (\nabla u, \nabla)_Q = (f, v)_Q \qquad \forall\, v \in L^2(I; H_0^1(\Omega)),$$

with $f \in C(\overline{I}; L^2(\Omega))$ given. In the implicit Euler discretisation of the (skew-symmetric) Nitsche method, we look for a discrete function $u_{h,\tau} \in \mathbb{P}^0(\mathcal{J}_\tau; V_h)$ that solves

$$\sum_{j=1}^{m_\tau} (u_{h,\tau}(t_j) - u_{h,\tau}(t_{j-1}), v_{h,\tau}(t_j))_\Omega + (\nabla u_h, \nabla v_h)_Q - (\nabla u_h \cdot \boldsymbol{n}, v_h)_{I\times\partial\Omega}$$
$$+ (\nabla v_h \cdot \boldsymbol{n}, u_h)_{I\times\partial\Omega} + \alpha(h_{\mathcal{F}}^{-1} u_h, v_h)_{I\times\partial\Omega} = (f_\tau, v_h)_Q$$

for all $v_{h,\tau} \in \mathbb{P}^0(\mathcal{J}_\tau; V_h)$, where $u_{h,\tau}(0)$ is set as the L^2-projection of a given initial datum $u_0 \in L^2(\Omega)$, and $\alpha > 0$ is a parameter to be chosen. (As in the lecture, f_τ is the zeroth order Lagrange interpolant on $\mathcal{J}_\tau$).

(a) Prove that a discrete solution exists, and derive appropriate a priori error estimates, analogous to those for conforming space approximations from the lecture.

Hint: You will also need the mesh-dependent norm from Sheet 1.

(b) Apply the discrete Aubin–Lions lemma (Theorem 2.4.2) to prove that the sequence of discrete solutions $\{u_{h,\tau}\}_{h,\tau}$ is precompact in $L^2(Q)$.

(c) **[Bonus +5 points.]** Prove that $u_{h,\tau}$ converges *strongly* in $L^2(I; H^1(\Omega))$ to the exact solution u.

Task 2 (Nonlinear heat transfer) (2+2+2 Points)

Consider the Laplace equation with a nonlinear boundary condition arising from radiative heat transfer:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\nabla u \cdot \mathbf{n} = |u|^{q-2} u & \text{on } \partial\Omega. \end{cases}$$

(a) Write down a weak formulation for this system for a given $f \in H^{-1}(\Omega)$; you have to choose an appropriate range of $q > 1$ so that the integrals make sense.

(b) Propose a conforming finite element discretisation with shape-regular triangulations and, assuming that discrete solutions exist (we will prove this in the lecture), derive appropriate uniform bounds.

(c) Prove convergence of the finite element discretisation.

(d) **[Bonus +4 points.]** Try the same for the parabolic version of this system.