Numerical Analysis for Nonlinear PDE

Summer Semester 2026 —— Sheet 2

Task 1 (More strong convergence) (2+3 Points)

Take the sequence of functionals I_n on $H^1_0(\Omega)$ from Theorem 3.2.2 and their Γ-limit I.

(a) Prove the claim we made about strong convergence: if $u_n \rightharpoonup u$ weakly in $H^1_0(\Omega)$ and $I_n(u_n) \to I(u)$, then $u_n \to u$ strongly in $H^1_0(\Omega)$.

(b) Suppose that V_n are finite element spaces of degree at least 1 with maximal mesh size h_n, and that $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Prove that we then obtain the convergence rate for the minimisers:

$$\|u_n - u\|_{H^1_0(\Omega)} \leq c h_n^2 \tag{1}$$

(c) **[Bonus]** Could you prove similar results for the Nitsche formulation from Sheet 1, Problem 3(d)?

Task 2 (Strong measurability of semidiscrete functions) (3 Points)

Let V_n be a finite-dimensional subspace of a Banach space V with a basis $\{\varphi_j\}_{j=1}^n$, and let $\{\psi_j\}_{j=1}^n$ be functions in $L^1(I)$ with $I \subset \mathbb{R}$ an interval. Prove that the function $v\colon I \to V$ defined as

$$v(t) := \sum_{j=1}^{n} \psi_j(t)\varphi_j,$$

is strongly measurable.

Task 3 (General forcing terms) (4 Points)

In the analysis of the fully discrete approximation in Section 4.2 of the notes, we assumed that the load $f \in C(I; H^{-1}(\Omega))$ was continuous in time. For general loads in the dual of the energy space $f \in L^2(I; H^{-1}(\Omega))$, we could define the piecewise-constant (in time) approximations $f_k\colon I \to H^{-1}(\Omega)$ through:

$$f_k(t) := f_k^j := \frac{1}{\tau}\int_{I_j} f(s)\,\mathrm{d}s \qquad \text{for } t \in I_j := (t_{j-1}, t_j],\ j \in \{1, \ldots, m_k\}, \tag{2}$$

and use f_k instead of f; this is well-defined, since point values of f_k are well-defined (as opposed to those of f). Provide a justification for why basically the same proof would yield convergence to a weak solution with load f (similar remarks can be made about $A(t)$).

Task 4 (Summation-by-parts) (2+2 Points)

Take two finite sequences $\{v_j\}_{j=0}^k, \{w_j\}_{j=0}^k$ of elements of a Hilbert space H, and denote the backward difference quotient (with a given time step $\tau > 0$) by

$$\mathrm{d}_\tau v_j := \frac{v_j - v_{j-1}}{\tau} \qquad \text{for } j \in \{1, \ldots, k\}. \tag{3}$$

(a) Prove the discrete product rule $\mathrm{d}_\tau (v_j, w_j)_H = (\mathrm{d}_\tau v_j, w_j)_H + (v_{j-1}, \mathrm{d}_\tau w_j)$, for $j \in \{1, \ldots, k\}$.

(b) Prove the Summation-by-parts formula:

$$\tau \sum_{j=1}^{k} \left[(\, \mathrm{d}_\tau v_j, w_j)_H + (v_{j-1},\, \mathrm{d}_\tau w_j)_H \right] = (v_k, w_k)_H - (v_0, w_0). \tag{4}$$