Numerical Analysis for Nonlinear PDE

Summer Semester 2026 —— Sheet 10

Task 1 (The obstacle problem) (3+3+3+3+4 Points)

This task will focus on the obstacle problem posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, which consists in finding a function $u \in H_0^1(\Omega)$ that minimises the following energy

$$I\colon H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$$
$$I(v) := \tfrac{1}{2}\|\nabla v\|_\Omega^2 - \langle f, v\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} + \chi_K(v),$$

where $f \in H^{-1}(\Omega)$ is given, and K is the constraint set associated to a given obstacle $g \in H^1(\Omega)$ with $g|_{\partial\Omega} \leq 0$:

$$K := \{v \in H_0^1(\Omega) \mid v \geq g \text{ a.e. on } \Omega\}.$$

(a) Prove that I is convex, proper and lower semicontinuous on $H_0^1(\Omega)$.

(b) Using the direct method of the calculus of variations, prove that there is a unique solution $u \in H_0^1(\Omega)$ to the obstacle problem, which can be characterised as the solution to the following variational inequality:

$$\int_\Omega \nabla u \cdot \nabla(v - u) \geq \langle f, v - u\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \qquad \forall\, v \in K.$$

(c) Assuming additional regularity, e.g. $u \in C^2(\overline{\Omega})$, $\varphi, f \in C(\overline{\Omega})$ should suffice, prove that the obstacle problem can be re-written as:

$$\begin{cases} -\Delta u - f \geq 0 & \text{in } \Omega,\\ u - \varphi \geq 0 & \text{in } \Omega,\\ (u-\varphi)(\Delta u + f) = 0 & \text{in } \Omega. \end{cases}$$

How can you interpret these conditions?

(d) Prove that $\mu := -\Delta u - f \in H^{-1}(\Omega)$ is positive and therefore can be identified with a Radon measure.

Hint: Use Theorem 2.1.4.

(e) Derive a dual problem.

(f) [**Bonus**] It can be shown that every $v \in H_0^1(\Omega)$ has a representative $\widetilde{v}$ that is μ-measurable for which one can write

$$\langle \mu, v\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = \int_\Omega \widetilde{v}\, \mathrm{d}\mu.$$

Moreover, the obstacle constraint can be interpreted also as $\widetilde{u} \geq \widetilde{g}$ μ-a.e. in Ω. In addition, there is a sequence $\{g_k\}_{k\in\mathbb{N}} \subset K$ such that $\widetilde{g}_k$ decreases pointwise μ-a.e. to $\widetilde{g}$.

Using these facts, prove that $u \in K$ is the solution to the obstacle problem if and only if $\mu := -\Delta u - f \in H^{-1}(\Omega)$ is a positive Radon measure and the following complementarity condition holds:

$$\int_\Omega (\widetilde{u} - \widetilde{g})\, \mathrm{d}\mu = 0$$

This is the generalisation of (c) for less regular solutions.