Numerical Analysis for Nonlinear PDE
Summer Semester 2026 —— Sheet 1

Task 1 (Zarantonello iteration for strongly monotone problems)   (3 Points)

Let V be a Hilbert space and let $A\colon V \to V^*$ be strongly monotone and Lipschitz continuous; i.e. there are two constants $m, M > 0$ such that for any $v_1, v_2 \in V$ one has:

$$\langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V \geq m\|v_1 - v_2\|_V^2, \tag{1a}$$
$$\|A(v_1) - A(v_2)\|_{V^*} \leq M\|v_1 - v_2\|_V. \tag{1b}$$

Show that there is a constant $\theta > 0$ such that the sequence $\{u_k\}_{k \in \mathbb{N}} \subset V$ generated by:

$$u_k = u_{k-1} - \theta J_V(A(u_{k-1}) - f), \tag{2}$$

where $f \in V^*$ is given, $u_0 \in V$ chosen arbitrarily, and $J_V\colon V^* \to V$ is the Riesz map, converges to a limit $u \in V$ that satisfies $A(u) = f$

Hint: Banach's fixed point theorem.

[**Bonus +3 Points**] Consider the Zarantonello iteration (2) on a sequence of finite dimensional subspaces $\{V_n\}_{n \in \mathbb{N}}$ of V such that $\overline{\bigcup_n V_n} = V$. Prove that the sequence of discrete solutions $u_k^n \in V_n$ converges to u in V.

Task 2 (Strong convergence)   (2+2 Points)

In the proof of Theorem 3.1.1 in the notes, we saw that the weakly converging Galerkin approximations $u_n \rightharpoonup u$ satisfy:

$$\lim_{n \to \infty} \langle A(u_n), u_n \rangle_{H_0^1(\Omega)} = \langle A(u), u \rangle_{H_0^1(\Omega)}, \tag{3}$$

where $A\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined for all $v, w \in H_0^1(\Omega)$ as $\langle A(v), w \rangle_{H_0^1(\Omega)} := \int_\Omega \underline{a}\nabla v \cdot \nabla w$.

(a) Prove the claim we made: the condition (3) implies that actually $u_n \to u$ *strongly* in $H_0^1(\Omega)$.

(b) Prove that the strong convergence still holds for strongly monotone and Lipschitz A, with the weaker condition

$$\limsup_{n \to \infty} \langle A(u_n), u_n \rangle_{H_0^1(\Omega)} \leq \langle A(u), u \rangle_{H_0^1(\Omega)}, \tag{4}$$

Task 3 (Nitsche's method)   (3+3+3 Points)

Take a family of shape-regular triangulations $\{\mathcal{T}_h\}_{h>0}$ and associated finite element spaces $V_h \subset H^1(\Omega)$, as described in Section 2.6 of the lecture notes. Consider the Laplace equation:

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \qquad \forall \in H_0^1(\Omega),$$

with $f \in L^2(\Omega)$. In the *Nitsche method* for the weak imposition of boundary values, we look for a discrete function $u_h \in V_h$ that solves

$$(\nabla u_h, \nabla v_h)_\Omega - (\nabla u_h \cdot \boldsymbol{n}, v_h)_{\partial\Omega} - \theta(\nabla v_h \cdot \boldsymbol{n}, u_h)_{\partial\Omega} + \alpha(h_{\mathcal{F}}^{-1} u_h, v_h)_{\partial\Omega} = (f, v_h)_\Omega,$$

for all $v_h \in V_h$; here $\theta \in \{-1, 0, 1\}$, and $\alpha > 0$ is a parameter to be chosen. Also, $h_{\mathcal{F}} := \partial\Omega \to (0, \infty)$ is the piecewise constant function with $h_{\mathcal{F}}|_F = h_F$ for a boundary facet $F \in \mathcal{F}_h^\partial$. (Note that u_h does not belong to the target space $H_0^1(\Omega)$.)

(a) Prove that for large enough $\alpha > 0$, the discrete solution $u_h \in V_h$ exists, and it is uniformly bounded in the following h-dependent norm

$$\|v_h\|_h := \left(\|\nabla v_h\|_\Omega^2 + \left\|h_{\mathcal{F}}^{-\frac{1}{2}} v_h\right\|_{\partial\Omega}^2 \right)^{1/2}.$$

Hint: use the inverse facet-to-element inequality.

(b) Prove that u_h converges weakly in $H^1(\Omega)$ to a limit $u \in H_0^1(\Omega)$, and that this limit is in fact the solution of the Laplace equation.

(c) For the skew-symmetric variant of the Nitsche method ($\theta = -1$), prove that the convergence of u_h is in fact strong.

(d) [**Bonus +2 points.**] Prove that the symmetric variant ($\theta = 1$) can also be interpreted as the optimality condition $J_h'(u_h) = 0$ in V_h^*, for the minimisation of an appropriately defined energy $J_h \colon V_h \to \mathbb{R}$.

(e) [**Bonus +2 points**] Prove that the discrete variational problems from (d) Γ-converge to the usual Dirichlet principle for the Laplace problem.