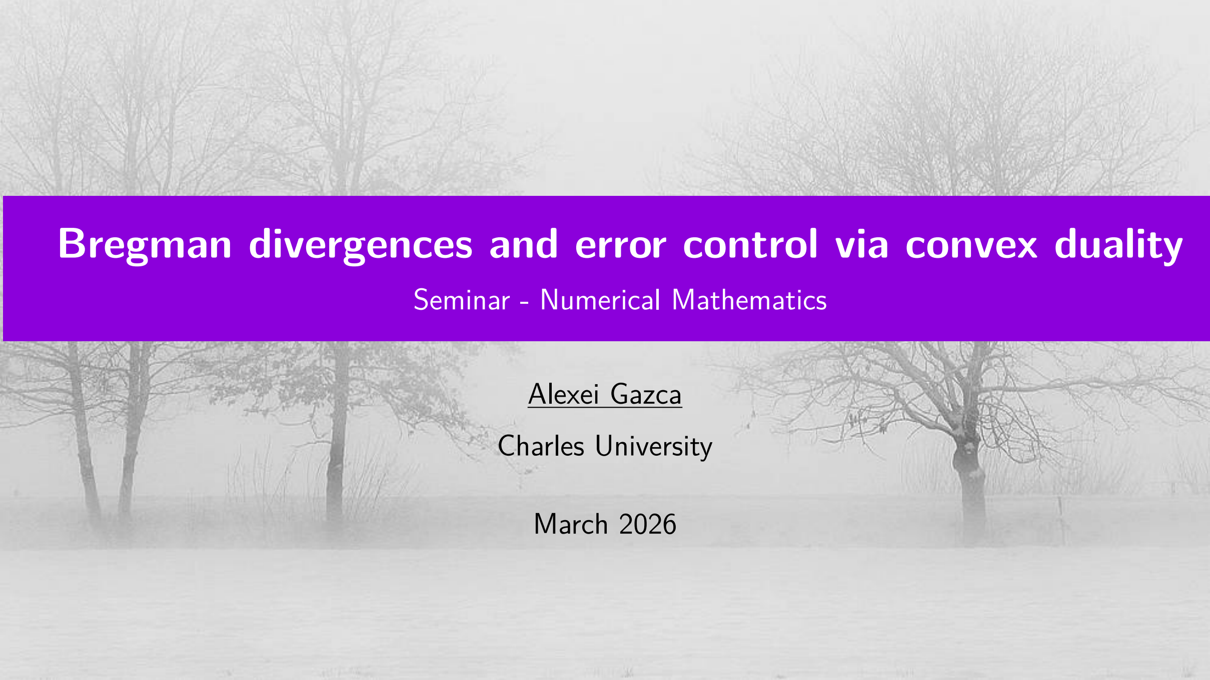

Bregman divergences and error control via convex duality

Seminar - Numerical Mathematics

Alexei Gazca

Charles University

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Joint work with:

Alex Kaltenbach (TU Berlin)

- 📕 P.A. Gazca–Orozco, A. Kaltenbach *A Priori and A Posteriori Error Identities for Vectorial Problems via Convex Duality*. ArXiv Preprint: 2602.04368, 2026.
- 📕 P.A. Gazca–Orozco *Bregman divergences and error control via convex duality.*. In preparation.

2 The Laplace Equation

Let Ω be Lipschitz, $f \in L^2(\Omega)$

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Find $u \in H_0^1(\Omega)$ s.t.:

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The solution $u \in H_0^1(\Omega)$ is a minimiser of the convex energy $I \colon H_0^1(\Omega) \to \mathbb{R}$

$$I(u) = \min_{v \in H_0^1(\Omega)} I(v)$$

$$I(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v = \frac{1}{2} \|\nabla v\|_\Omega^2 - (f, v)_\Omega$$

2 The Laplace Equation (mixed I)

Introduce the flux $\mathbf{q} = \nabla u \in L^2(\Omega)^d$. One can then equivalently write:

$$\begin{cases} -\operatorname{div}\mathbf{q} = f & \text{in } \Omega, \\ \mathbf{q} = \nabla u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Find $(\mathbf{q}, u) \in L^2(\Omega)^d \times H^1_0(\Omega)$ s.t.:

$$\begin{aligned} -(\mathbf{q}, \mathbf{r})_\Omega + (\nabla u, \mathbf{r})_\Omega &= 0 && \forall\, \mathbf{r} \in L^2(\Omega)^d \\ (\mathbf{q}, \nabla v)_\Omega &= (f, v)_\Omega && \forall\, v \in H^1_0(\Omega). \end{aligned}$$

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The solution $(\mathbf{q}, u)$ is a saddle point of the Lagrangian:

$$\max_{\mathbf{r}\in L^2(\Omega)} \mathcal{L}(u, \mathbf{r}) = \mathcal{L}(\mathbf{q}, u) = \min_{v \in H_0^1(\Omega)} \mathcal{L}(\mathbf{q}, v)$$

$$\mathcal{L}(\mathbf{r}, v) := -\tfrac{1}{2}\|\mathbf{r}\|_\Omega^2 + (\mathbf{r}, \nabla v)_\Omega - (f, v)_\Omega$$

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The primal energy can be written in terms of the Lagrangian:

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Lagrangian theory can help derive dual (maximisation) problems:

$$D(\mathbf{q}) = \sup_{\mathbf{r} \in L^2(\Omega)} D(\mathbf{r})$$

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In general there can be a duality gap:

$$D(\mathbf{r}) \leq D(\mathbf{q}) = \sup_{\mathbf{r} \in L^2(\Omega)} \inf_{v \in H^1_0(\Omega)} \mathcal{L}(\mathbf{r}, v) \leq \inf_{v \in H^1_0(\Omega)} \sup_{\mathbf{r} \in L^2(\Omega)} \mathcal{L}(\mathbf{r}, v) = I(u) \leq I(v)$$

2 Lagrangian duality

In general, the following are equivalent:

- $(\mathbf{q}, u)$ is a saddle point of $\mathcal{L}$
- u is a solution of the primal problem, $\mathbf{q}$ is a solution of the dual problem, there is no duality gap, i.e. $I(u) = D(\mathbf{q})$

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For Laplace, computing $\inf_{v \in H^1_0(\Omega)} \mathcal{L}(\mathbf{r}, v) = -\frac{1}{2} \|\mathbf{r}\|^2_\Omega + (\mathbf{r}, \nabla v)_\Omega - (f, v)_\Omega$ leads to

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This implies that

$$\mathbf{r} \in H(\mathrm{div}\,; \Omega) := \{\mathbf{s} \in L^2(\Omega)^d \mid \mathrm{div}\, \mathbf{s} \in L^2(\Omega)\}$$
$$\mathrm{div}\, \mathbf{r} = -f$$

2 The Laplace equation (dual)

The dual problem corresponds to the maximisation problem:

$$D(\mathbf{q}) = \sup_{\mathbf{r} \in L^2(\Omega)} D(\mathbf{r})$$

$$D(\mathbf{r}) := -\tfrac{1}{2} \|\mathbf{r}\|_\Omega^2 - I_{\{-f\}}^\Omega(\operatorname{div} \mathbf{r})$$

The characteristic function $I_{\{-f\}}^\Omega$ is interpreted as:

$$I_{\{-f\}}^\Omega(\operatorname{div} \mathbf{r}) := \begin{cases} 0 & \text{if } \operatorname{div} \mathbf{r} = -f \\ +\infty & \text{otherwise.} \end{cases}$$

Note that we could maximise over $\mathbf{r} \in H(\operatorname{div}; \Omega)$.

2 The Laplace equation (mixed II)

Note that we could maximise over $\mathbf{r} \in H(\mathrm{div}\,;\Omega)$, and used the Lagrangian $\widetilde{\mathcal{L}} \colon H(\mathrm{div}\,,\Omega) \times L^2(\Omega) \to \mathbb{R}$:

$$\widetilde{\mathcal{L}}(\mathbf{r}, v) := -\tfrac{1}{2}\,\|\mathbf{r}\|_\Omega^2 - (\mathrm{div}\,\mathbf{r}, v)_\Omega - (f, v)_\Omega$$

This leads to the optimality conditions:

$$\begin{aligned} \partial_1 \widetilde{\mathcal{L}}(\mathbf{q}, u) &= 0 & -(\mathbf{q}, \mathbf{r})_\Omega - (\mathrm{div}\,\mathbf{r}, u)_\Omega &= 0 & \forall\, \mathbf{r} &\in H(\mathrm{div}\,;\Omega) \\ \partial_2 \widetilde{\mathcal{L}}(\mathbf{q}, u) &= 0 & -(\mathrm{div}\,\mathbf{q}, v) &= (f, v)_\Omega & \forall\, v &\in L^2(\Omega). \end{aligned}$$

The first condition implies that $\nabla u = \mathbf{q} \in L^2(\Omega)$, and so $u \in H_0^1(\Omega)$.

2 The Prager-Synge identity

The Prager-Synge (Hypercircle) Theorem [Prager, Synge; '47]

Let $v \in H_0^1(\Omega)$ and $\mathbf{r} \in H(\operatorname{div};\Omega)$ with $\operatorname{div}\mathbf{r} = -f$ be arbitrary. Then

$$\frac{1}{2}\|\nabla u - \nabla v\|_\Omega^2 + \frac{1}{2}\|\mathbf{q} - \mathbf{r}\|_\Omega^2 = \frac{1}{2}\|\nabla v - \mathbf{r}\|_\Omega^2$$

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- The solution is characterised by three properties:
  - $u \in H^1_0(\Omega)$;
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- The RHS is computable $\Longrightarrow$ a posteriori estimation [Ainsworth, Oden, Braess, Schöberl, Bartels, Carstensen, Ern, Vohralík...].
- Geometric proof (Pythagoras) $\Longrightarrow$ difficult to generalise

2 Convex conjugates

The (convex) conjugate of a (proper) function $\varphi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the (proper, convex, lsc) function $\varphi^*\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$\varphi^*(\mathbf{a}^*) := \sup_{\mathbf{a}\in\mathbb{R}^n} \left[\mathbf{a}^* \cdot \mathbf{a} - \varphi(\mathbf{a})\right].$$

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For any proper convex lsc $\varphi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the Fenchel–Young inequality holds:

$$\mathbf{a}^* \cdot \mathbf{a} \le \varphi(\mathbf{a}) + \varphi^*(\mathbf{a}^*) \qquad \forall\, \mathbf{a}, \mathbf{a}^* \in \mathbb{R}^n,$$

with equality being equivalent to the duality relations:

$$\mathbf{a}^* \cdot \mathbf{a} = \varphi(\mathbf{a}) + \varphi^*(\mathbf{a}^*) \quad \Leftrightarrow \quad \mathbf{a} = \nabla\varphi^*(\mathbf{a}^*) \quad \Leftrightarrow \quad \mathbf{a}^* = \nabla\varphi(\mathbf{a}).$$

(Or $\mathbf{a} \in \partial\varphi^*(\mathbf{a}^*) \Leftrightarrow \mathbf{a}^* \in \partial\varphi(\mathbf{a})$ for non-smooth φ.)

2 The generalised Prager-Synge identity

This can be generalised to convex energies on $W^{1,p}(\Omega)$:

$$I(v) := \int_\Omega [\varphi(\cdot, \nabla v) + \psi(\cdot, v)] \qquad D(\mathbf{r}) = -\int_\Omega [\varphi^*(\cdot, \mathbf{r}) + \psi^*(\cdot, \operatorname{div} \mathbf{r})]$$

- S. Repin *A posteriori estimates for partial differential equations*. Radon Series on Computational and Applied Mathematics 4, Walter de Gruyter GmbH & Co. KG, 2008.
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Define the total error:

$$\rho^2_{\mathrm{tot}}(v, \mathbf{r}) := \rho^2_I(v) + \rho^2_{-D}(\mathbf{r}) := I(v) - I(u) + D(\mathbf{q}) - D(\mathbf{r})$$

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Thanks to strong duality, one concludes:

$$\rho^2_{\mathrm{tot}}(v, \mathbf{r}) = I(v) - D(\mathbf{r}) =: \eta^2_{\mathrm{gap}}(v, \mathbf{r})$$

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So the gap estimator is

$$\eta^2_{\mathrm{gap}}(v, \mathbf{r}) = \int_\Omega [\varphi(\cdot, \nabla v) - \mathbf{r} \cdot \nabla v + \varphi^*(\cdot, \mathbf{r})] + \int_\Omega [\psi(\cdot, v) - \operatorname{div} \mathbf{r}\, v + \psi^*(\cdot, \operatorname{div} \mathbf{r})]$$

- It is an identity $\Longrightarrow$ reliability and efficiency with constants one.

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- It is an identity ⟹ reliability and efficiency with constants one.
- Globally equivalent to residual-type estimators.
- The integrands are non-negative and vanish if and only if the optimality conditions are satisfied.
- They can be localised ⟹ adaptive mesh refinement

2 The Prager-Synge identity (Laplace)

For Laplace, Taylor implies:

$$\rho_I^2(v) = I(v) - I(u) = \frac{1}{2} I''(u)[v - u, v - u] = \frac{1}{2} \left\| \nabla v - \nabla u \right\|_\Omega^2$$

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For the dual energy:

$$\rho^2_{-D}(\mathbf{r}) = D(\mathbf{q}) - D(\mathbf{r}) = \frac{1}{2}\left\|\mathbf{r}\right\|_\Omega^2 - \frac{1}{2}\left\|\mathbf{q}\right\|_\Omega^2$$

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&= \tfrac{1}{2} \|\mathbf{r} - \mathbf{q}\|_\Omega^2
\end{aligned}$$

So

$$\rho_{\mathrm{tot}}^2(v, \mathbf{r}) = \tfrac{1}{2} \|\nabla v - \nabla u\|_\Omega^2 + \tfrac{1}{2} \|\mathbf{r} - \mathbf{q}\|_\Omega^2$$

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The duality gap estimator takes the form:

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$$\rho^2_{\mathrm{tot}}(v,\mathbf{r}) = \eta^2_{\mathrm{gap}}(v,\mathbf{r}) \quad \Longleftrightarrow \quad \tfrac{1}{2}\,\|\nabla u - \nabla v\|^2_\Omega + \tfrac{1}{2}\,\|\mathbf{q} - \mathbf{r}\|^2_\Omega = \tfrac{1}{2}\,\|\nabla v - \mathbf{r}\|^2_\Omega$$

2 Local efficiency

The error estimator can be localised:

$$\tfrac{1}{2}\left\|\nabla v - \mathbf{r}\right\|_{\Omega}^{2} = \sum_{K \in \mathcal{T}_h} \eta_K^2(\nabla v, \mathbf{r}) \qquad \eta_K^2(\nabla v, \mathbf{r}) := \frac{1}{2}\int_K |\nabla v - \mathbf{r}|^2.$$

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For Laplace this is just the triangle inequality:

$$\begin{aligned} \eta_K^2(\nabla v, \mathbf{r}) &\le \tfrac{1}{2}\left(\|\nabla v - \nabla u\|_K + \|\mathbf{q} - \mathbf{r}\|_K\right)^2 \\ &\le \|\nabla v - \nabla u\|_K^2 + \|\mathbf{q} - \mathbf{r}\|_K^2 \\ &= 2\mathcal{E}_K \end{aligned}$$

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We will achieve this with the help of Bregman divergences:

$$\mathcal{D}_\varphi(\mathbf{a}, \mathbf{b}) := \varphi(\mathbf{a}) - \varphi(\mathbf{b}) - \nabla\varphi(\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}).$$

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(or use a subgradient $\mathbf{r} \in \partial\varphi(\mathbf{b})$ in the non-smooth case.)

This is a generalised notion of distance:

- Not necessarily symmetric $\mathcal{D}_\varphi(\mathbf{a}, \mathbf{b}) \neq \mathcal{D}_\varphi(\mathbf{b}, \mathbf{a})$.
- $\mathcal{D}_\varphi(\mathbf{a}, \mathbf{b}) \geq 0$ and $\mathcal{D}_\varphi(\mathbf{a}, \mathbf{a}) = 0$.
- If φ is strictly convex: $\mathcal{D}_\varphi(\mathbf{a}, \mathbf{b}) > 0$ if $\mathbf{a} \neq \mathbf{b}$.
- If $\mathbf{p} = \nabla\varphi(\mathbf{b})$, then $\mathcal{D}_\varphi(\mathbf{a}, \mathbf{b}) = \varphi(\mathbf{a}) + \varphi^*(\mathbf{p}) - \mathbf{p} \cdot \mathbf{a}$.
- For $\varphi(\mathbf{a}) = \frac{1}{2}|\mathbf{a}|^2$ we have $\mathcal{D}_\varphi(\mathbf{a}, \mathbf{b}) = \frac{1}{2}|\mathbf{a} - \mathbf{b}|^2$.

2 Updated error measures

We will replace our error measures:

$$\begin{aligned} I(v) - I(u) \qquad &\mapsto \qquad \rho_I^2(v) := \int_\Omega \mathcal{D}_\varphi(\nabla v, \nabla u) + \int_\Omega \mathcal{D}_\psi(v, u) \\ D(\mathbf{q}) - D(\mathbf{r}) \qquad &\mapsto \qquad \rho_{-D}^2(\mathbf{r}) := \int_\Omega \mathcal{D}_{\varphi^*}(\mathbf{r}, \mathbf{q}) + \int_\Omega \mathcal{D}_{\psi^*}(\operatorname{div} \mathbf{r}, \operatorname{div} \mathbf{q}) \end{aligned}$$

The integrands are now pointwise non-negative and vanish if and only if $(v, \mathbf{r}) = (u, \mathbf{q})$.

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$$D(\mathbf{q}) - D(\mathbf{r}) \qquad \mapsto \qquad \rho_{-D}^2(\mathbf{r}) := \int_\Omega \mathcal{D}_{\varphi^*}(\mathbf{r}, \mathbf{q}) + \int_\Omega \mathcal{D}_{\psi^*}(\operatorname{div} \mathbf{r}, \operatorname{div} \mathbf{q})$$

The integrands are now pointwise non-negative and vanish if and only if $(v, \mathbf{r}) = (u, \mathbf{q})$.

Proposition [A. G; 2026]

Suppose strong duality $I(u) = D(\mathbf{q})$ is satisfied. Then the following generalised Prager-Synge identity holds:

$$\rho_I^2(v) + \rho_{-D}^2(\mathbf{r}) = \sum_{K \in \mathcal{T}_h} \eta_K^2(v, \mathbf{r})$$

$$\eta_K^2(v, \mathbf{r}) := \int_K [\varphi(\cdot, \nabla v) - \mathbf{r} \cdot \nabla v + \varphi^*(\cdot, \mathbf{r})] + \int_K [\psi(\cdot, v) - \operatorname{div} \mathbf{r}\, v + \psi^*(\cdot, \operatorname{div} \mathbf{r})],$$

2 Local efficiency

Proposition [A. G.; 2026]

Suppose that the Bregman–Young inequality holds (here $\mathbf{r}_2 \in \partial\varphi(\mathbf{a}_2)$):

$$|(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2)| \leq c_{\mathrm{BY}} \left(\mathcal{D}_\varphi(\mathbf{a}_1, \mathbf{a}_2) + \mathcal{D}_{\varphi^*}(\mathbf{r}_1, \mathbf{r}_2)\right).$$

Then the duality gap estimator is locally efficient:

$$\eta^2_{\varphi;K}(\nabla v, \mathbf{r}) \lesssim \int_K \left[\mathcal{D}_\varphi(\nabla v, \nabla u) + \mathcal{D}_{\varphi^*}(\mathbf{r}, \mathbf{q})\right],$$

where the constant depends only on c_{BY}.

The assumptions will be satisfied e.g. if $\varphi(\cdot, \mathbf{a}) = \widehat{\varphi}(|\mathbf{a}|)$, where $\widehat{\varphi}$ is a uniformly convex N-function.

2 Uniformly convex N-functions

A function $\widehat{\varphi}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is an N-function if [Diening, Ettwein, Růžička, Fornasier...]

- $\widehat{\varphi}$ is continuous and convex;
- There is a right-continuous, non-decreasing $\widehat{\varphi}'\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with
  - $\widehat{\varphi}(t) = \int_0^t \widehat{\varphi}'(s)\mathrm{d}s$
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  - $\widehat{\varphi}'(s) > 0$ for $s > 0$
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In particular $\lim_{t\to 0+} \frac{\widehat{\varphi}(t)}{t} = 0$ and $\lim_{t\to +\infty} \frac{\widehat{\varphi}(t)}{t} = +\infty$.

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The N-function $\widehat{\varphi}$ is said to be uniformly convex if

$$c_{\mathrm{uc}} \frac{\widehat{\varphi}'(s) - \widehat{\varphi}'(t)}{s-t} \leq \frac{\widehat{\varphi}'(s)}{s} \leq C_{\mathrm{uc}} \frac{\widehat{\varphi}'(s) - \widehat{\varphi}'(t)}{s-t}$$

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Some examples:

$$\widehat{\varphi}(t) = \frac{t^p}{p} \qquad \widehat{\varphi}(t) = t(\log(1+t))^p \qquad p \in (1,\infty).$$

2 The φ-Laplacian

Consider the energy

$$I(v) := \int_\Omega \widehat{\varphi}(|\nabla v|) - \int_\Omega fv,$$

with uniformly convex $\widehat{\varphi}$. In this case there is a unique minimiser u, belonging to the Orlicz–Sobolev space:

$$W_0^{1,\varphi}(\Omega) := \left\{ v \in W_0^{1,1}(\Omega) \mid \nabla v \in L^\varphi(\Omega) \right\}$$

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The minimiser u satisfies

$$\int_\Omega \mathcal{A}(\nabla u) \cdot \nabla v = \int_\Omega fv \qquad \forall\, v \in W_0^{1,\varphi}(\Omega).$$

with $\mathcal{A}(\nabla u) := \frac{\widehat{\varphi}'(|\nabla u|)}{|\nabla u|} \nabla u$ (With $\widehat{\varphi}(t) = \frac{t^p}{p}$ one has $\mathcal{A}(\nabla u) = |\nabla u|^{p-2} \nabla u$)

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$$\|v - u\|_{W^{1,p}(K)} ,$$

but this is suboptimal.

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- Local efficiency for the p-Laplacian in terms of the natural distance was known [Bartels, Kaltenbach; 2024], but the estimate contains non-explicit constants.
- Our error measure $\mathcal{D}_\varphi(\nabla v, \nabla u)$ is equivalent to the natural distance, and the estimate depends only on $c_{\mathrm{uc}}, C_{\mathrm{uc}}$.

2 Estimates for non-conforming functions

- So far things work with conforming functions $v \in W^{1,1}(\Omega)$, but in practice non-conforming methods can be useful.

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  Problem: If $v \notin W^{1,1}(\Omega)$, then $I(v) = +\infty$...
- Define an extended energy on the broken Sobolev space $W^{1,1}(\mathcal{T}_h)$:

$$\bar{I}\colon W^{1,1}(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$$
$$\bar{I}(v) := \int_\Omega \varphi(\cdot, \mathfrak{G}_h v) + \int_\Omega \psi(\cdot, v),$$

  where $\mathfrak{G}_h$ is a discrete approximation of the gradient (piecewise gradient, DG gradient...)

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- Similarly for the dual energy:

$$\overline{D}(\mathbf{r}) := -\int_\Omega \varphi^*(\cdot, \mathbf{r}) - \int_\Omega \psi^*(\cdot, \mathfrak{Div}_h \mathbf{r}),$$

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- Of course, with strong duality one still has:

$$\overline{I}(v) - I(u) + D(\mathbf{q}) - \overline{D}(\mathbf{r}) = \overline{I}(v) - \overline{D}(\mathbf{r}),$$

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  but $\overline{I}(v) - I(u)$ and $D(\mathbf{q}) - \overline{D}(\mathbf{r})$ are not necessarily non-negative, so cannot represent errors!
- Solution: Use Bregman divergences:

$$\overline{\rho}_{\overline{I}}(v) := \int_\Omega \mathcal{D}_\varphi(\mathfrak{G}_h v, \nabla u) + \int_\Omega \mathcal{D}_\psi(v, u)$$

$$\overline{\rho}_{-\overline{D}}(\mathbf{r}) := \int_\Omega \mathcal{D}_{\varphi^*}(\mathbf{r}, \mathbf{q}) + \int_\Omega \mathcal{D}_{\psi^*}(\mathfrak{Div}_h \mathbf{r}, \operatorname{div} \mathbf{q})$$

2 Estimates for non-conforming functions

- We have the error relation

$$\overline{\rho}_{\overline{I}}(v)+\overline{\rho}_{-\overline{D}}(\mathbf{r}) = \sum_{K\in\mathcal{T}_h} \eta_K^2(v,\mathbf{r}) + (\mathbf{r}-\mathbf{q}, \mathfrak{G}_h v - \nabla u)_\Omega + (\mathfrak{Div}_h \mathbf{r} - \operatorname{div}\mathbf{q}, \mathfrak{G}_h v - \nabla u)_\Omega$$

- But note that $(\mathbf{q}, \nabla u)_\Omega + (\operatorname{div}\mathbf{q}, u)_\Omega = 0$, since $\mathbf{q}$ and u are conforming!

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- But note that $(\mathbf{q}, \nabla u)_\Omega + (\operatorname{div}\mathbf{q}, u)_\Omega = 0$, since $\mathbf{q}$ and u are conforming!
- Hence, if $\mathbf{r}$ is conforming $\Longrightarrow$ we can swap $u \mapsto s$ for an arbitrary conforming $s \in W^{1,1}(\Omega)$ in the remaining terms!
- We can then close the estimate with a Young-type inequality with ε.

2 Local efficiency

Proposition [A. G.; 2026]

Suppose that the Bregman–Young inequality with ε holds (here $\mathbf{r}_2 \in \partial\varphi(\mathbf{a}_2)$):

$$|(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_3)| \lesssim \varepsilon \left(\mathcal{D}_\varphi(\mathbf{a}_1, \mathbf{a}_2) + \mathcal{D}_{\varphi^*}(\mathbf{r}_1, \mathbf{r}_2)\right) + \mathcal{D}_\varphi(\mathbf{r}_1, \mathbf{r}_3).$$

(Similarly for ψ).

If $\mathbf{r}$ is div-conforming, then we have the a posteriori bound:

$$\overline{\rho}_{\overline{I}}(v) + \overline{\rho}_{-\overline{D}}(\mathbf{r}) \lesssim \sum_{K \in \mathcal{T}_h} \eta_K^2(v, \mathbf{r}) + \inf_{s \in W_0^{1,1}(\Omega)} \left[\int_\Omega \mathcal{D}_\varphi(\mathfrak{G}_h v, \nabla s) + \int_\Omega \mathcal{D}_\psi(v, s)\right]$$

A similar statement holds if v is conforming.

- Uniformly convex integrands satisfy the assumptions.

2 Key properties

Crucial properties that we'd like to preserve at the discrete level:

- Poincaré inequality:

$$\|v\|_\Omega \lesssim \|\nabla v\|_\Omega \qquad \forall v \in H_0^1(\Omega)$$

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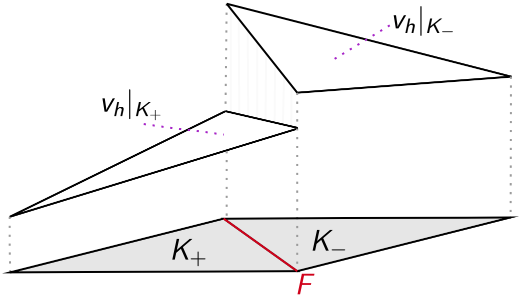
- Orthogonality $\Longrightarrow$ $H(\operatorname{div}; \Omega)$ regularity

$$\mathbf{r} \in L^2(\Omega)^d, \quad (\nabla v, \mathbf{r})_\Omega = 0 \quad \forall v \in H_0^1(\Omega) \quad \Longrightarrow \quad \mathbf{r} \in H(\operatorname{div}; \Omega),\ \operatorname{div} \mathbf{r} = 0$$

2 Piecewise polynomials

On a shape regular mesh, denote the space of piecewise polynomials of degree k as:

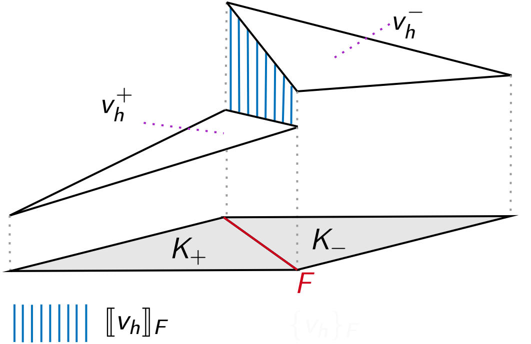
$$\mathbb{P}_k(\mathcal{T}_h) := \{v_h \in L^\infty(\Omega) \mid v_h|_K \in \mathbb{P}_k(K), \text{ for all } K \in \mathcal{T}_h\}$$



2 Jumps and averages

Define the jump of $v_h \in \mathbb{P}_k(\mathcal{T}_h)$ on a facet $F \in \mathcal{F}_h$ as:

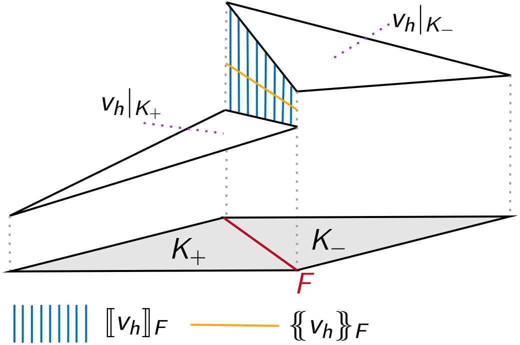
$$[\![v_h]\!]_F := \begin{cases} v_h^+|_F - v_h^-|_F & \text{if } F \in \mathcal{F}_h^i,\ F = K_+ \cap K_- \\ v_h|_F & \text{if } F \in \mathcal{F}_h^\partial. \end{cases}$$



2 Jumps and averages

Define the average of $v_h \in \mathbb{P}_k(\mathcal{T}_h)$ on a facet $F \in \mathcal{F}_h$ as:

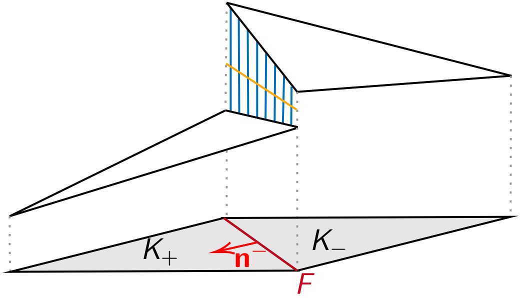
$$\{\!\{v_h\}\!\}_F := \begin{cases} \frac{1}{2}\left(v_h^+|_F + v_h^-|_F\right) & \text{if } F \in \mathcal{F}_h^i,\ F = K_+ \cap K_- \\ v_h|_F & \text{if } F \in \mathcal{F}_h^\partial. \end{cases}$$



2 Jumps and averages

For a vector-valued function $\mathbf{r}_h \in \mathbb{P}_k(\mathcal{T}_h)^d$:

$$[\![\mathbf{r}_h \cdot \mathbf{n}]\!]_F := \begin{cases} \mathbf{r}_h^+ \cdot \mathbf{n}^+ + \mathbf{r}_h^- \cdot \mathbf{n}^- & \text{if } F \in \mathcal{F}_h^i,\ F = K_+ \cap K_- \\ \mathbf{r}_h|_F & \text{if } F \in \mathcal{F}_h^\partial. \end{cases}$$

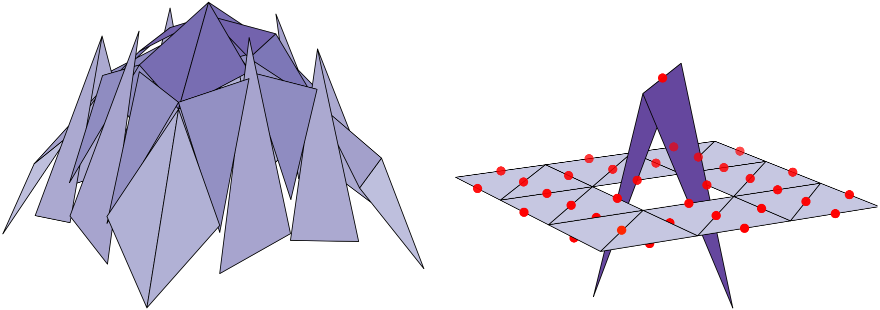


|||||||| $[\![v_h]\!]_S$ —— $\{\!\{v_h\}\!\}_F$

2 Crouzeix–Raviart element

$$V^h := \left\{ v_h \in \mathbb{P}_1(\mathcal{T}_h) \mid \fint_F [\![v_h]\!]_F = [\![v_h]\!]_F(x_F) = 0 \text{ for all } F \in \mathcal{F}_h^i \right\}$$

$$V_D^h := \left\{ v_h \in V^h \mid \fint_F v_h = v_h(x_F) = 0 \text{ for all } F \in \mathcal{F}_h^D \right\} \not\subset H_0^1(\Omega)$$



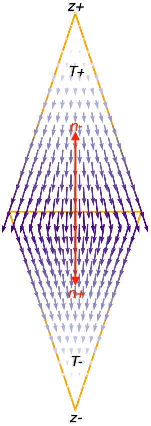
(a) Crouzeix–Raviart minimiser on $\Omega = (-1,1)^2$ with $f \equiv 1$

(b) Crouzeix–Raviart basis function

2 Raviart–Thomas element

$$\Sigma^h := \Big\{ \mathbf{r}_h \in \mathbb{P}_1(\mathcal{T}_h)^d \mid \mathbf{r}_h \cdot \mathbf{n} \in \mathbb{P}_0(F) \text{ for all } F \in \mathcal{F}_h, \\ [\![\mathbf{r}_h \cdot \mathbf{n}]\!]_F = 0 \text{ for all } F \in \mathcal{F}_h^i, \Big\}$$

$$\Sigma_N^h := \Big\{ \mathbf{r}_h \in \Sigma^h \mid \mathbf{r}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma_N \Big\}$$



(a) RT basis function

$$\Sigma^h \subset H(\operatorname{div}; \Omega)$$

$$\operatorname{div} \Sigma^h = \mathbb{P}_0(\mathcal{T}_h)$$

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- Poincaré inequality (if $|\Gamma_D| > 0$): $\|v_h\|_\Omega \lesssim \|\nabla_h v_h\|_\Omega \qquad \forall\, v_h \in V_D^h$

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- Poincaré inequality (if $|\Gamma_D| > 0$): $\|v_h\|_\Omega \lesssim \|\nabla_h v_h\|_\Omega \qquad \forall\, v_h \in V_D^h$
- Discrete integration by parts formula:

$$(\nabla_h v_h, \mathbf{r}_h)_\Omega + (v_h, \operatorname{div} \mathbf{r}_h)_\Omega = (v_h, \mathbf{r}_h \cdot \mathbf{n})_{\partial\Omega} \qquad \forall v_h \in V^h,\ \mathbf{r}_h \in \Sigma^h.$$

- Orthogonality between discrete gradients and divergence-free fields:

$$(\nabla_h v_h, \mathbf{r}_h)_\Omega = 0 \qquad \forall\, v \in V_D^h,\ \mathbf{r}_h \in \Sigma_N^h,\ \operatorname{div} \mathbf{r} = 0.$$

2 Key properties at the discrete level

- Poincaré inequality (if $|\Gamma_D| > 0$): $\|v_h\|_\Omega \lesssim \|\nabla_h v_h\|_\Omega \qquad \forall\, v_h \in V_D^h$
- Discrete integration by parts formula:

$$(\nabla_h v_h, \mathbf{r}_h)_\Omega + (v_h, \operatorname{div} \mathbf{r}_h)_\Omega = (v_h, \mathbf{r}_h \cdot \mathbf{n})_{\partial\Omega} \qquad \forall v_h \in V^h,\, \mathbf{r}_h \in \Sigma^h.$$

- Orthogonality between discrete gradients and divergence-free fields:

$$(\nabla_h v_h, \mathbf{r}_h)_\Omega = 0 \qquad \forall\, v \in V_D^h,\, \mathbf{r}_h \in \Sigma_N^h,\, \operatorname{div} \mathbf{r} = 0.$$

- Zero discrete divergence in $\mathbb{P}_0(\mathcal{T}_h)^d \implies$ Raviart–Thomas

$$\Pi_h \Sigma_N^h \perp_{L^2} \nabla_h (\ker \Pi_h|_{V_D^h})$$

$$(\mathbf{r}_h, \nabla_h v_h)_\Omega = 0 \quad \forall\, v_h \in V_D^h \text{ s.t. } \Pi_h v_h = 0 \quad \implies \quad \mathbf{r}_h = \Pi_h \tilde{\mathbf{r}}_h \text{ for some } \tilde{\mathbf{r}}_h \in \Sigma_N^h$$

S. Bartels, Z. Wang *Orthogonality relations of Crouzeix–Raviart and Raviart–Thomas finite element spaces*. Numer. Math. 148:127–139, 2021

2 Discrete strong duality

These properties allow the derivation of discrete strong duality principles for

$$I_h \colon V_D^h \to \mathbb{R} \cup \{+\infty\} \qquad\qquad D_h \colon \Sigma_N^h \to \mathbb{R} \cup \{+\infty\}$$

$$I_h(v_h) := \int_\Omega \varphi_h(\cdot, \nabla_h v_h) + \int_\Omega \psi_h(\cdot, \Pi_h v_h) \qquad\qquad D_h(\mathbf{r}_h) = \int_\Omega \varphi_h^*(\cdot, \Pi_h \mathbf{r}_h) + \psi_h^*(\cdot, \operatorname{div} \mathbf{r}_h)$$

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The canonical discretisations would otherwise be:

- Mixed I: $(u, \mathbf{q}) \in H_D^1(\Omega) \times L^2(\Omega)^d \mapsto (u_h, \mathbf{q}_h) \in V_D^h \times \mathbb{P}_0(\mathcal{T}_h)^d$
- Mixed II: $(u, \mathbf{q}) \in L^2(\Omega) \times H_N(\operatorname{div}; \Omega)^d \mapsto (u_h, \mathbf{q}_h) \in \mathbb{P}_0(\mathcal{T}_h) \times \Sigma_N^h$

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In contrast, the duality approach delivers:

- Information about gradients: H^1-convergence for u_h
- Physically conforming flux: $\mathbf{q} \in H(\operatorname{div}; \Omega)$

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It is possible to write optimality relations pointwise:

$$\begin{aligned} \Pi_h \mathbf{q}_h &= D\varphi_h(\cdot, \nabla_h u_h) && \text{a.e. in } \Omega \\ \operatorname{div} \mathbf{q}_h &= D\psi_h(\cdot, u_h) && \text{a.e. in } \Omega \end{aligned}$$

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Crucially, one can obtain generalised Marini formulas:

$$\mathbf{q}_h = D\varphi_h(\cdot, \nabla_h u_h) + \frac{D\psi_h(\cdot, \Pi_h u_h)}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})$$

2 A priori error analysis (Laplace)

For $f \in L^2(\Omega)$, find $u_h \in V_D^h$ s.t.

$$a_h(u_h, v_h) := (\nabla_h u_h, \nabla_h v_h) = (f, v_h)_\Omega$$

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Classical analysis [Berger, Scott, Strang; 1972]:

- Extend the bilinear form a_h to $H_0^1(\Omega) + V_D^h$
- With the help of an elliptic projection, prove the error bound

$$\|\nabla u - \nabla_h u_h\|_\Omega \leq \inf_{v_h \in V_D^h} \|\nabla u - \nabla v_h\|_\Omega + \sup_{v_h \in V_D^h} \frac{a_h(u - u_h, v_h)}{\|\nabla_h v_h\|_\Omega}$$

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- To handle the second term, note that

$$a_h(u - u_h, v_h) = -\sum_{F \in \mathcal{F}_h^i} \int_F \nabla u \cdot \mathbf{n}_F \, [\![v]\!]_F$$

This requires $u \in H^{3/2+\varepsilon}(\Omega)$...

2 A priori error analysis (Laplace)

Alterantive approach [Ern, Guermond; 2021]:

$$\|u - u_h\|_\sharp \leq c \inf_{v_h \in V_D^h} \|u - v_h\|_\sharp \qquad \|v_h\|_\sharp := \sum_{K \in \mathcal{T}_h} \left(\|\nabla v\|_\Omega^2 + h_K \|\nabla v|_K \cdot \mathbf{n}\|_{\partial K}^2 \right)$$

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“Minimal regularity” estimates for $u \in H^1(\Omega)$ [Gudi; 2010]:

$$\|\nabla u - \nabla_h u_h\|_\Omega \leq c \left(\inf_{v_h \in V_h} \|\nabla u - \nabla_h v_h\|_\Omega + \operatorname{osc}(f) \right)$$

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Truly minimal regularity ($u \in H_0^1(\Omega)$, $f \in H^{-1}(\Omega)$) estimates [Zanotti, Veeser; 2019]:

$$\|\nabla u - \nabla_h u_h\| \sim \inf_{v_h \in V_h} \|\nabla u - \nabla_h v_h\|_\Omega \qquad \langle f, v_h \rangle_\Omega \mapsto \langle f, E_h v \rangle_\Omega$$

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What is c?

2 A priori error analysis via duality (Laplace)

Convex duality easily delivers constant-free and (almost) minimal-regularity ($f \in L^2(\Omega)$) estimates [Bartels, Kaltenbach; 2024]:

$$\tfrac{1}{2}\,\|\nabla_h v_h - \nabla_h u_h\|_\Omega^2 + \tfrac{1}{2}\,\|\mathbf{r}_h - \mathbf{q}_h\|_\Omega^2 \leq \tfrac{1}{2}\,\|\mathbf{q}_h - \mathbf{r}_h\|_\Omega^2$$

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New idea: Mimic the a posteriori arguments (Prager–Synge) at the discrete level.

$$\text{Estimate } (u, \mathbf{q}) - (v, \mathbf{r}) \longrightarrow \text{Estimate } (u_h, \mathbf{q}_h) - (v_h, \mathbf{r}_h)$$

→ recent work on the Signorini problem [Bartels, Gudi, Kaltenbach; 2024].

2 A priori error analysis via duality (Laplace)

$$I_h(v_h) := \frac{1}{2}\,\|\nabla_h v_h\|_\Omega^2 - (f_h, v_h)_\Omega \qquad D_h(\mathbf{r}_h) = -\frac{1}{2}\,\|\Pi_h \mathbf{r}_h\|_\Omega^2 - I_{\{-f_h\}}^\Omega(\operatorname{div}\mathbf{r}_h)$$

- Step 1: identify the strong convexity measures:

$$\rho_{I_h}^2(v_h) := I_h(v_h) - I_h(u_h) = \frac{1}{2}\,\|\nabla_h v_h - \nabla_h u_h\|_\Omega^2$$
$$\rho_{-D_h}^2(\mathbf{r}_h) := D_h(\mathbf{q}_h) - D_h(\mathbf{r}_h) = \frac{1}{2}\,\|\Pi_h \mathbf{r}_h - \Pi_h \mathbf{q}_h\|_\Omega^2$$

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- **Step 2:** examine the discrete gap estimator:

$$\eta_{\mathrm{gap},h}^2(v_h, \mathbf{r}_h) := I_h(v_h) - D_h(\mathbf{r}_h) = \frac{1}{2}\,\|\nabla_h v_h - \mathbf{r}_h\|_\Omega^2$$

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- Step 3: use the discrete strong duality $I_h(u_h) = D_h(\mathbf{q}_h)$:

$$\rho^2_{\mathrm{tot},h}(v_h, \mathbf{r}_h) = \eta^2_{\mathrm{gap},h}(v_h, \mathbf{r}_h)$$
$$\frac{1}{2}\|\nabla_h u_h - \nabla_h v_h\|_\Omega^2 + \frac{1}{2}\|\Pi_h \mathbf{q}_h - \Pi_h \mathbf{r}_h\|_\Omega^2 = \frac{1}{2}\|\nabla_h v_h - \Pi_h \mathbf{r}_h\|_\Omega^2$$

2 A priori error analysis via duality (Laplace)

Set $v_h = \mathcal{I}_h^{cr}(u)$ and $\mathbf{r}_h = \mathcal{I}_h^{rt}(\mathbf{q})$:

$$\|\nabla_h \mathcal{I}_h^{cr} u - \nabla_h u_h\|_\Omega^2 + \|\Pi_h \mathcal{I}_h^{rt} \mathbf{q} - \Pi_h \mathbf{q}_h\|_\Omega^2 = \|\nabla_h \mathcal{I}_h^{cr} u - \Pi_h \mathcal{I}_h^{rt} \mathbf{q}\|_\Omega^2$$

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Using the commuting property $\nabla_h \mathcal{I}_h^{cr} u = \Pi_h \nabla u = \Pi_h \mathbf{q}$, we obtain an a priori error identity:

$$\boxed{\|\nabla_h \mathcal{I}_h^{cr} u - \nabla_h u_h\|_\Omega^2 + \|\Pi_h \mathcal{I}_h^{rt}\mathbf{q} - \Pi_h \mathbf{q}_h\|_\Omega^2 = \|\Pi_h(\mathbf{q} - \mathcal{I}_h^{rt}\mathbf{q})\|_\Omega^2}$$

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Similarly, we get quasioptimality with explicit constants:

$$\|\nabla u - \nabla_h u_h\|_\Omega + \|\mathbf{q} - \Pi_h \mathbf{q}_h\|_\Omega \sim \inf_{v_h \in V_D^h} \|\nabla u - \nabla_h v_h\|_\Omega + \inf_{\substack{\mathbf{r}_h \in \Sigma^h \\ \operatorname{div} \mathbf{r}_h = -f_h}} \|\mathbf{q} - \Pi_h \mathbf{r}_h\|_\Omega$$

2 A priori error analysis via duality (φ-Laplace)

Take a discretisation of the φ-Laplace problem:

$$I_h(v_h) := \int_\Omega \varphi(\nabla_h v_h) - (f_h, v_h)_\Omega - (F_h, \nabla_h v_h)_\Omega - (g_h, v_h)_{\Gamma_N},$$

$$D_h(\mathbf{r}_h) := -\int_\Omega \varphi^*(\Pi_h \mathbf{r}_h) - \chi^{\Omega}_{\{-f_h\}}(\operatorname{div}(\mathbf{r}_h - \mathbf{F}_h)) - \chi^{\Gamma_N}_{\{g_h\}}((\mathbf{r}_h - \mathbf{F}_h)\mathbf{n})$$

Here f_h, F_h, g_h are piecewise constant approximations of load terms $f, F, g \in L^{\varphi^*}$.

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Error bounds for Crouzeix–Raviart discretisations:

- [Yan, Liu, Carstensen...] Pioneering works with restrictive assumptions.
- [Kaltenbach; 2024], [Storn; 2025] *Medius analysis*: includes oscillation terms and shape-regularity-dependent constants.
- [Blechta, G-O, Kaltenbach, Růžička; 2026] Requires jump stabilisation and implementing a smoothing operator.
- [Diening, Hirn, Kreuzer, Zanotti; 2025] Focuses on Stokes (pressure-robustness), requires a smoothing operator and shape-regularity-dependent constants.

2 A priori error analysis via duality (φ-Laplace)

Proposition [A. G.; 2026]

The discrete generalised Prager–Synge identity holds for all $v_h \in V_D^h$ and $\mathbf{r}_h \in K_h$:

$$\int_\Omega \mathcal{D}_\varphi(\nabla_h u_h, \nabla u) + \int_\Omega \mathcal{D}_{\varphi^*}(\Pi_h \mathbf{q}_h, \mathbf{q}) = \int_\Omega [\varphi(\nabla_h v_h) - \Pi_h \mathbf{r}_h \cdot \nabla_h v_h + \varphi^*(\Pi_h \mathbf{r}_h)] \,.$$

Furthermore, quasioptimality holds:

$$\int_\Omega \mathcal{D}_\varphi(\nabla_h u_h, \nabla u) + \int_\Omega \mathcal{D}_{\varphi^*}(\Pi_h \mathbf{q}_h, \mathbf{q}) \sim \inf_{v_h \in V_D^h} \int_\Omega \mathcal{D}_\varphi(\nabla_h v_h, \nabla u) + \inf_{\mathbf{r}_h \in K_h} \int_\Omega \mathcal{D}_{\varphi^*}(\Pi_h \mathbf{r}_h, \mathbf{q}),$$

where the constants depend only on $c_{\mathrm{uc}}, C_{\mathrm{uc}}$.

2 Other results

We get similar results for:

- Laplace (new for duality-based analysis):
  - quasioptimality with known constants and minimal regularity
  - Right-hand-side $f \in H^1(\Omega)$.
  - Mixed inhomogeneous Dirichlet–Neumann BCs.
- Incompressible Stokes:
  - A posteriori identity (known with geometric proof)
  - Extended energy ⇒ Estimate for non-divergence-free velocities (known with geometric proof).
  - Stress reconstruction formula (known with restrictive assumptions)
  - Quasioptimality with known constants and minimal regularity (new).
- Linear elasticity:
  - A posteriori identity (known with geometric proof)
  - Extended energy ⇒ Estimate for non-symmetric stresses (known with geometric proof).
  - Stress reconstruction formula (new)
  - A priori estimate with known constants and minimal regularity (new).

Thank you for your attention!