A priori and a posteriori estimates for vectorial problems via convex duality

Equadiff 2026

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June 2026

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GRANTOVÁ AGENTURA ČESKÉ REPUBLIKY

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1 The Laplace Equation

$$\text{Find } u \in H_0^1(\Omega): \qquad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall\, v \in H_0^1(\Omega).$$

- The solution $u \in H_0^1(\Omega)$ is a minimiser of the convex energy

$$I(u) = \min_{v \in H_0^1(\Omega)} I(v) := \tfrac{1}{2} \|\nabla v\|_\Omega^2 - (f, v)_\Omega$$

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- The flux $\mathbf{q} = \nabla u \in L^2(\Omega)^d$ is the solution to the dual problem

$$D(\mathbf{q}) = \sup_{\mathbf{r} \in L^2(\Omega)} D(\mathbf{r}) := -\tfrac{1}{2} \|\mathbf{r}\|^2_\Omega - I^\Omega_{\{-f\}}(\operatorname{div} \mathbf{r})$$

- Strong duality holds: $I(u) = D(\mathbf{q})$

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1 The Prager-Synge identity

The Prager-Synge (Hypercircle) Theorem [Prager, Synge; '47]

Let $v \in H_0^1(\Omega)$ and $\mathbf{r} \in H(\operatorname{div}; \Omega)$ with $\operatorname{div} \mathbf{r} = -f$ be arbitrary. Then

$$\frac{1}{2} \|\nabla u - \nabla v\|_\Omega^2 + \frac{1}{2} \|\mathbf{q} - \mathbf{r}\|_\Omega^2 = \frac{1}{2} \|\nabla v - \mathbf{r}\|_\Omega^2$$

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- The RHS is computable $\Longrightarrow$ a posteriori estimation [Ainsworth, Oden, Braess, Schöberl, Bartels, Carstensen, Ern, Vohralík...].

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- The RHS is computable $\Longrightarrow$ a posteriori estimation [Ainsworth, Oden, Braess, Schöberl, Bartels, Carstensen, Ern, Vohralík...].
- Geometric proof (Pythagoras) $\Longrightarrow$ not obvious how to generalise

1 The generalised Prager-Synge identity

This can be generalised to convex energies on $W^{1,p}(\Omega)$:

$$I(v) := \int_\Omega \phi(\cdot, \nabla v) + \int_\Omega \psi(\cdot, v) \qquad D(\mathbf{r}) = \int_\Omega \phi^*(\cdot, \mathbf{r}) + \psi^*(\cdot, \operatorname{div} \mathbf{r})$$

- S. Repin *A posteriori estimates for partial differential equations*. Radon Series on Computational and Applied Mathematics 4, Walter de Gruyter GmbH & Co. KG, 2008.
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Define the total error:

$$\rho^2_{\mathrm{tot}}(v, \mathbf{r}) := \rho^2_I(v) + \rho^2_{-D}(\mathbf{r}) := I(v) - I(u) + D(\mathbf{q}) - D(\mathbf{r})$$

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Thanks to strong duality, one concludes:

$$\rho^2_{\mathrm{tot}}(v, \mathbf{r}) = I(v) - D(\mathbf{r}) =: \eta^2_{\mathrm{gap}}(v, \mathbf{r})$$

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1 Key properties

Crucial properties that we'd like to preserve at the discrete level:

- Poincaré inequality:

$$\|v\|_\Omega \lesssim \|\nabla v\|_\Omega \qquad \forall v \in H_0^1(\Omega)$$

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$$(\nabla v, \mathbf{r})_\Omega = -(v, \operatorname{div} \mathbf{r})_\Omega \qquad \forall\, v \in H_0^1(\Omega),\ \mathbf{r} \in H(\operatorname{div}; \Omega).$$

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- Orthogonality between gradients and divergence-free fields:

$$(\nabla v, \mathbf{r})_\Omega = 0 \qquad \forall\, v \in H_0^1(\Omega),\ \mathbf{r} \in H(\operatorname{div}; \Omega),\ \operatorname{div} \mathbf{r} = 0.$$

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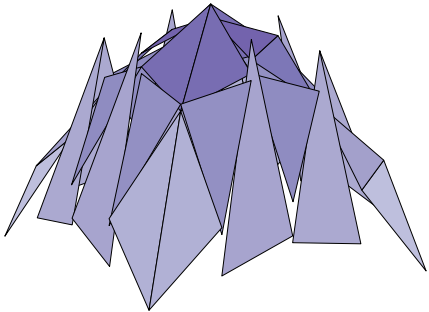
- Orthogonality $\Longrightarrow$ $H(\operatorname{div}; \Omega)$ regularity

$$\mathbf{r} \in L^2(\Omega)^d, \quad (\nabla v, \mathbf{r})_\Omega = 0 \quad \forall v \in H_0^1(\Omega) \quad \Longrightarrow \quad \mathbf{r} \in H(\operatorname{div}; \Omega),\, \operatorname{div} \mathbf{r} = 0$$

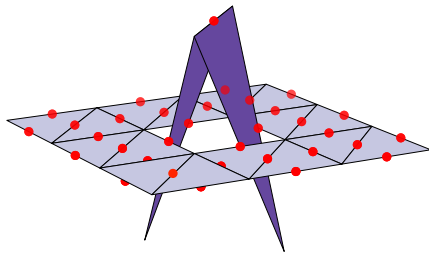
1 Crouzeix–Raviart element

$$V^h := \left\{ v_h \in \mathbb{P}_1(\mathcal{T}_h) \mid \fint_F [\![v_h]\!]_F = [\![v_h]\!]_F(x_F) = 0 \text{ for all } F \in \mathcal{F}_h^i \right\}$$

$$V_D^h := \left\{ v_h \in V^h \mid \fint_F v_h = v_h(x_F) = 0 \text{ for all } F \in \mathcal{F}_h^D \right\} \not\subset H_0^1(\Omega)$$



(a) Crouzeix–Raviart minimiser on $\Omega = (-1, 1)^2$ with $f \equiv 1$



(b) Crouzeix–Raviart basis function

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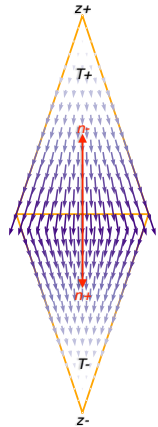
1 Raviart–Thomas element

$$\Sigma^h := \Big\{ \mathbf{r}_h \in \mathbb{P}_1(\mathcal{T}_h)^d \mid \mathbf{r}_h \cdot \mathbf{n} \in \mathbb{P}_0(F) \text{ for all } F \in \mathcal{F}_h, \quad [\![\mathbf{r}_h \cdot \mathbf{n}]\!]_F = 0 \text{ for all } F \in \mathcal{F}_h^i, \Big\}$$

$$\Sigma_N^h := \Big\{ \mathbf{r}_h \in \Sigma^h \mid \mathbf{r}_h \cdot \mathbf{n} = 0 \text{ on } \Gamma_N \Big\}$$

$$\Sigma^h \subset H(\mathrm{div}\,; \Omega)$$

$$\mathrm{div}\, \Sigma^h = \mathbb{P}_0(\mathcal{T}_h)$$



(a) RT basis function

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1 Key properties at the discrete level

- Poincaré inequality (if $|\Gamma_D| > 0$): $\|v_h\|_\Omega \lesssim \|\nabla_h v_h\|_\Omega \qquad \forall\, v_h \in V_D^h$

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- Poincaré inequality (if $|\Gamma_D| > 0$): $\|v_h\|_\Omega \lesssim \|\nabla_h v_h\|_\Omega \qquad \forall\, v_h \in V_D^h$
- Discrete integration by parts formula:

$$(\nabla_h v_h, \mathbf{r}_h)_\Omega + (v_h, \operatorname{div} \mathbf{r}_h)_\Omega = (v_h, \mathbf{r}_h \cdot \mathbf{n})_{\partial\Omega} \qquad \forall v_h \in V^h,\ \mathbf{r}_h \in \Sigma^h.$$

- Orthogonality between discrete gradients and divergence-free fields:

$$(\nabla_h v_h, \mathbf{r}_h)_\Omega = 0 \qquad \forall\, v \in V_D^h,\ \mathbf{r}_h \in \Sigma_N^h,\ \operatorname{div} \mathbf{r} = 0.$$

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- Zero discrete divergence in $\mathbb{P}_0(\mathcal{T}_h)^d \implies$ Raviart–Thomas

$$\Pi_h \Sigma_N^h \perp_{L^2} \nabla_h(\ker \Pi_h|_{V_D^h})$$

$$(\mathbf{r}_h, \nabla_h v_h)_\Omega = 0 \quad \forall\, v_h \in V_D^h \text{ s.t. } \Pi_h v_h = 0 \quad \implies \quad \mathbf{r}_h = \Pi_h \widetilde{\mathbf{r}_h} \text{ for some } \widetilde{\mathbf{r}_h} \in \Sigma_N^h$$

S. Bartels, Z. Wang *Orthogonality relations of Crouzeix–Raviart and Raviart–Thomas finite element spaces*. Numer. Math. 148:127–139, 2021

These properties allow the derivation of discrete strong duality principles for

$$I_h \colon V_D^h \to \mathbb{R} \cup \{+\infty\} \qquad D_h \colon \Sigma_N^h \to \mathbb{R} \cup \{+\infty\}$$

$$I_h(v_h) \coloneqq \int_\Omega \phi_h(\cdot, \nabla_h v_h) + \int_\Omega \psi_h(\cdot, \Pi_h v_h) \qquad D_h(\mathbf{r}_h) = \int_\Omega \phi_h^*(\cdot, \Pi_h \mathbf{r}_h) + \psi_h^*(\cdot, \operatorname{div} \mathbf{r}_h)$$

It is possible to write optimality relations pointwise; e.g. for Laplace

$$\Pi_h \mathbf{q}_h = \nabla_h u_h \qquad \text{a.e. in } \Omega$$
$$\operatorname{div} \mathbf{q}_h = -\Pi_h f \qquad \text{a.e. in } \Omega$$

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Crucially, one can derive a formula for the dual solution:

$$\mathbf{q}_h = \nabla_h u_h - \frac{f_h}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})$$

1 A priori error analysis (Laplace)

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$$\|\nabla u - \nabla_h u_h\|_\Omega \leq c \left(\inf_{v_h \in V_h} \|\nabla u - \nabla_h v_h\|_\Omega + \operatorname{osc}(f) \right)$$

$$\operatorname{osc}(f) := \sum_{K \in \mathcal{T}_h} \left(h_K^2 \|f - f_h\|_K^2 \right)$$

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- Truly minimal regularity ($u \in H_0^1(\Omega)$, $f \in H^{-1}(\Omega)$) estimates [Zanotti, Veeser; 2019]:

$$\|\nabla u - \nabla_h u_h\| \sim \inf_{v_h \in V_h} \|\nabla u - \nabla_h v_h\|_\Omega \qquad \langle f, v_h \rangle_\Omega \mapsto \langle f, E_h v \rangle_\Omega$$

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What is c?

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New idea: Mimic the a posteriori arguments (Prager–Synge) at the discrete level.

$$\text{Estimate } (u, \mathbf{q}) - (v, \mathbf{r}) \longrightarrow \text{Estimate } (u_h, \mathbf{q}_h) - (v_h, \mathbf{r}_h)$$

$$\rho^2_{\mathrm{tot},h}(v_h, \mathbf{r}_h) = \eta^2_{\mathrm{gap},h}(v_h, \mathbf{r}_h)$$
$$\tfrac{1}{2}\,\|\nabla_h u_h - \nabla_h v_h\|^2_\Omega + \tfrac{1}{2}\,\|\Pi_h \mathbf{q}_h - \Pi_h \mathbf{r}_h\|^2_\Omega = \tfrac{1}{2}\,\|\nabla_h v_h - \Pi_h \mathbf{r}_h\|^2_\Omega$$

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$$\tfrac{1}{2}\left\|\nabla_h u_h - \nabla_h v_h\right\|^2_\Omega + \tfrac{1}{2}\left\|\Pi_h \mathbf{q}_h - \Pi_h \mathbf{r}_h\right\|^2_\Omega = \tfrac{1}{2}\left\|\nabla_h v_h - \Pi_h \mathbf{r}_h\right\|^2_\Omega$$

This way, we can obtain minimal regularity quasioptimality with explicit constants:

$$\left\|\nabla u - \nabla_h u_h\right\|_\Omega + \left\|\mathbf{q} - \Pi_h \mathbf{q}_h\right\|_\Omega \sim \inf_{v_h \in V_D^h} \left\|\nabla u - \nabla_h v_h\right\|_\Omega + \inf_{\substack{\mathbf{r}_h \in \Sigma^h \\ \operatorname{div} \mathbf{r}_h = -f_h}} \left\|\mathbf{q} - \Pi_h \mathbf{r}_h\right\|_\Omega$$

- Quasioptimality for the φ-Laplacian: [A.G.; 2026]

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1 Improvements

- Virtually all existing works within this framework deal with scalar problems with homogeneous Dirichlet BCs and $f \in L^2(\Omega)$.

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- Virtually all existing works within this framework deal with scalar problems with homogeneous Dirichlet BCs and $f \in L^2(\Omega)$.
- Inhomogeneous Dirichlet BC $u_D \in H^{1/2}(\Gamma_D)$:

$$\text{Find } u \in H^1_D(\Omega): \qquad (\nabla u, \nabla v)_\Omega = (f, v)_\Omega - (\nabla u_D, \nabla v)_\Omega \qquad \forall\, v \in H^1_D(\Omega).$$

  The desired solution is then $u_\star = u + u_D$.
- General load $\widehat{f} \in H_D^{-1}(\Omega)$; use the representation formula:

$$\langle \widehat{f}, v \rangle_\Omega = (f, v)_\Omega + (F, \nabla v)_\Omega \quad f \in L^2(\Omega),\ F \in L^2(\Omega)^d$$

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$$\begin{cases} -\nu\Delta\mathbf{u}_\star + \nabla p = \mathbf{f} - \operatorname{div}\mathbf{F} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u}_\star = 0 & \text{in } \Omega, \\ \mathbf{u}_\star = \mathbf{u}_D & \text{on } \Gamma_D, \\ (\nu\nabla\mathbf{u}_\star - p\mathbf{I})\mathbf{n} = \mathbf{F}\mathbf{n} & \text{on } \Gamma_N \end{cases}$$

Find $\mathbf{u} \in H^1_D(\Omega)^d$ with $\operatorname{div}\mathbf{u} = 0$, s.t.:

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{F}, \nabla\mathbf{v})_\Omega - (\nabla\mathbf{u}_D, \nabla\mathbf{v})_\Omega \quad \forall\, \mathbf{v} \in H^1_D(\Omega)^d,\ \operatorname{div}\mathbf{v} = 0.$$

The primal and dual problems are:

$$\min_{\mathbf{v}\in H^1_D(\Omega)^d} I(\mathbf{v}) := \tfrac{\nu}{2}\left\|\nabla(\mathbf{v}+\mathbf{u}_D)\right\|^2_\Omega + I^\Omega_{\{0\}}(\operatorname{div}\mathbf{v}) - \langle\widehat{\mathbf{f}}, \mathbf{v}\rangle_\Omega$$

$$\max_{\boldsymbol{\tau}\in\mathbf{F}+H(\operatorname{div};\Omega)^d} -\tfrac{1}{2\nu}\left\|\operatorname{dev}\boldsymbol{\tau}\right\|^2_\Omega + (\operatorname{dev}\boldsymbol{\tau}, \nabla\mathbf{u}_D) - I^\Omega_{\{-\mathbf{f}\}}(\operatorname{div}(\boldsymbol{\tau}-\mathbf{F})) - I^{\Gamma_N}_{\{0\}}((\boldsymbol{\tau}-\mathbf{F})\cdot\mathbf{n})$$

Here $\operatorname{dev}\boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}\operatorname{tr}\boldsymbol{\tau}\mathbf{I}$.

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- A posteriori error identity for general BCs and RHS
- Stress reconstruction formula

$$\mathbf{T}_h = \nu \nabla_h(\mathbf{u}_h + \mathbf{u}_D^h) - p_h \mathbf{I} - \tfrac{1}{d}\mathbf{f}_h \otimes (\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})$$

> Such a formula was known for $d = 2$, $\mathbf{f} \in L^2(\Omega)^d$, and Dirichlet BCs with datum $\mathbf{u}_D \in H^1(\mathcal{F}_h^\partial)^2 \cap H^1(\Omega)^2$ [Carstensen, Gallistl, Schedensack; 2013].

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- A priori error identity $\Longrightarrow$ minimal regularity quasioptimality with explicit constants
  - [Verfürth, Zanotti; 2019]: homogeneous Dirichlet BCs, smoothing operator $< \widehat{\mathbf{f}}, E_h \mathbf{v}_h >$, pressure-robust.

$$\begin{cases} -\operatorname{div} \mathbb{C}\mathbb{e}(\mathbf{u}_\star) = \mathbf{f} - \operatorname{div} \mathbf{F} & \text{in } \Omega, \\ \mathbf{u}_\star = \mathbf{u}_D & \text{on } \Gamma_D, \\ (\mathbb{C}\mathbb{e}(\mathbf{u}_\star))\mathbf{n} = \mathbf{F}\mathbf{n} & \text{on } \Gamma_N \end{cases}$$

Find $\mathbf{u} \in H^1_D(\Omega)^d$ s.t.:

$$(\mathbb{C}\mathbb{e}(\mathbf{u}), \mathbb{e}(\mathbf{v})) = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{F}, \nabla \mathbf{v})_\Omega - (\mathbb{e}(\mathbf{u}_D), \mathbb{e}(\mathbf{v}))_\Omega \quad \forall \mathbf{v} \in H^1_D(\Omega)^d$$

The primal and dual problems are:

$$\min_{\mathbf{v} \in H^1_D(\Omega)^d} I(\mathbf{v}) \coloneqq \tfrac{1}{2} \left\| \mathbb{C}^{1/2} \mathbb{e}(\mathbf{v} + \mathbf{u}_D) \right\|^2_\Omega - \langle \widehat{\mathbf{f}}, \mathbf{v} \rangle_\Omega$$

$$\max_{\boldsymbol{\tau} \in L^2(\Omega)^{d \times d}_{\mathrm{sym}}} D(\boldsymbol{\tau}) \coloneqq -\tfrac{1}{2} \left\| \mathbb{C}^{-1/2} \boldsymbol{\tau} \right\|^2_\Omega + (\boldsymbol{\tau}, \nabla \mathbf{u}_D) - I^\Omega_{\{-\mathbf{f}\}}(\operatorname{div}(\boldsymbol{\tau} - \mathbf{F})) - I^{\Gamma_N}_{\{0\}}((\boldsymbol{\tau} - \mathbf{F}) \cdot \mathbf{n})$$

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- By working with an extension $\overline{D}$ of D to non-symmetric tensors, we get estimates:

$$\tfrac{1}{2}\left\|\mathbb{C}^{\frac{1}{2}}\mathfrak{e}(\mathbf{v}_h-\mathbf{u})\right\|_\Omega^2+\tfrac{1}{2}\left\|\mathbb{C}^{-\frac{1}{2}}(\boldsymbol{\tau}_h-\boldsymbol{\sigma})\right\|_\Omega^2\sim\tfrac{1}{2}\left\|\mathbb{C}^{\frac{1}{2}}(\mathfrak{e}(\mathbf{v}_h+\mathbf{u}_D)-\mathbb{C}^{-1}\boldsymbol{\tau}_h)\right\|_\Omega^2,$$

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- We propose the reconstruction formula:

$$\boldsymbol{\sigma}_h=\mathbb{C}\mathfrak{e}(\mathbf{u}_h+\mathbf{u}_D^h)+\nabla_h r_h-\frac{2}{d+1}\mathrm{sym}\left(\mathbf{f}_h\otimes\mathrm{id}_{\mathbb{R}^d}-\Pi_h\mathrm{id}_{\mathbb{R}^d}\right)$$

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- No Korn inequality forces us to penalise jumps. Still, we prove minimal regularity (weak) quasioptimality (the constant depends on the discrete Korn constant)

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1 Perspectives

- Using convex duality we can obtain a priori and a posteriori error identities/estimates
  - A priori - minimal regularity $u \in H^1_D(\Omega)^d$, $f \in H^{-1}(\Omega)$
  - Inhomogeneous mixed BCs
  - Stokes and elasticity
- Marini reconstruction formula for the general setting.
  - Known for Stokes in particular cases
  - Seems new for elasticity
- A posteriori bounds for non-conforming functions.

For more details:

- P.A. Gazca–Orozco, A. Kaltenbach *A Priori and A Posteriori Error Identities for Vectorial Problems via Convex Duality*. ArXiv Preprint: 2602.04368, 2026.
- P.A. Gazca–Orozco *Bregman divergences and error control via convex duality.*. ArXiv Preprint: 2606.05088, 2026.

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Thank you for your attention!