Low-Mach consistency of a class of linearly implicit schemes for the compressible Euler equations

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WP 2.4 – Nonlinear convection-diffusion-reaction problems

Low-Mach flows

Compressible fluid flows

- Speed of sound $a = \sqrt{\gamma p/\rho}$.
- Determines the maximal speed at which information (usually) propagates in the flow.
- Mach number $M = v/a$.

Incompressible fluid flows

- Infinite speed of information propagation.
- Unphysical but useful model.
- As $M \to 0$, compressible $\to$ incompressible.

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$M \ll 1$ is challenging for numerics:

- On the border between compressible and incompressible.
- Explicit solvers: time step inversely proportional to maximal speed of information propagation - $\tau \approx Mh$.
- Implicit solvers: Condition number and properties of linear systems deteriorate as $M \to 0$.
- Acoustic waves become a severe problem.
- Numerical methods can produce incorrect solutions!!!

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Semi-implicit linearization

V. Dolejší, M. Feistauer, V. Kučera

$$\partial_t \boldsymbol{w} + \nabla\cdot \boldsymbol{f}(\boldsymbol{w}) = 0$$

- Homogeneity: $\boldsymbol{f}(\boldsymbol{w}) = \boldsymbol{f}'(\boldsymbol{w})\boldsymbol{w}$
- Semi-implicit scheme

$$\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{\Delta t} + \nabla\cdot \left(\boldsymbol{f}'(\boldsymbol{w}^n)\boldsymbol{w}^{n+1}\right) = 0.$$

- Discontinuous Galerkin method, non-reflecting boundary conditions, local artificial diffusion with shock capturing, preconditioning, ...

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Linearly implicit schemes

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Linearly implicit scheme based on a reference state

$$\frac{\boldsymbol{w}^{n+1} - \boldsymbol{w}^n}{\Delta t} + \nabla \cdot \Big(\boldsymbol{f}(\boldsymbol{w}^n) + \boldsymbol{f}'(\boldsymbol{w}_R^n)(\boldsymbol{w}^{n+1} - \boldsymbol{w}^n) \Big) = 0.$$

- For $\boldsymbol{w}_R^n =$ incompressible Euler, we get Kaiser et al.
- For $\boldsymbol{w}_R^n = \boldsymbol{w}^n$, we get Dolejší, Feistauer, Kučera.

Goal

Asymptotic consistency: We get the correct solution as $M = \varepsilon \to 0$.

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1. Low-Mach flows

2. Linearly implicit schemes

3. Asymptotic preserving analysis

Formal asymptotic analysis

Linearly implicit scheme

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Formal Hilbert expansion

Assume that $\rho, \boldsymbol{u}, E$ and p have an expansion of the form

$$\rho^n(x)=\rho^n_{(0)}(x)+\varepsilon\rho^n_{(1)}(x)+\varepsilon^2\rho^n_{(2)}(x)+O(\varepsilon^3).$$

- We expect e.g. that $\rho^n_{(0)}$ is constant for all n, similarly $\nabla\cdot\boldsymbol{u}^n_{(0)}=0$.
- Acoustics correspond to $O(\varepsilon)$ perturbations of $\rho, p, \nabla\cdot\boldsymbol{u}$.
- Plug in Hilbert expansions of all quantities into the scheme.
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Asymptotic preserving property

Theorem 1

Let the initial condition satisfy $\nabla\cdot \boldsymbol{u}^0_{(0)} = 0$ and $\rho^0_{(0)}$ be constant in space. Let the reference solution satisfy $\nabla\cdot \boldsymbol{u}^n_{R,(0)} = 0$ and $\rho^n_{R,(0)}$ be constant in space for all n. Assume either slip or periodic boundary conditions. Then for each n, $\rho^n_{(0)} = \rho^0_{(0)}$ and

$$\frac{\boldsymbol{u}^{n+1}_{(0)} - \boldsymbol{u}^n_{(0)}}{\Delta t} + \nabla\cdot\left(\boldsymbol{u}^{n+1}_{(0)} \otimes \boldsymbol{u}^{n+1}_{(0)}\right) + \nabla \frac{p^{n+1}_{(2)}}{\rho^{n+1}_{(0)}} = \mathcal{E}^{n+1},$$

$$\nabla\cdot \boldsymbol{u}^{n+1}_{(0)} = 0,$$

where $\mathcal{E}^{n+1}$ is a consistency error term satisfying

$$|\mathcal{E}^{n+1}| \leq C\|\boldsymbol{u}^{n+1}_{(0)} - \boldsymbol{u}^n_{(0)}\|_{W^{1,\infty}}\left(\|\boldsymbol{u}^{n+1}_{(0)} - \boldsymbol{u}^n_{(0)}\|_{W^{1,\infty}} + \|\boldsymbol{u}^n_{(0)} - \boldsymbol{u}^n_{R,(0)}\|_{W^{1,\infty}}\right),$$

where C depends only on γ.

Asymptotic preserving property

Consistency error

$$|\mathcal{E}^{n+1}| \leq C\|\boldsymbol{u}_{(0)}^{n+1} - \boldsymbol{u}_{(0)}^{n}\|_{W^{1,\infty}}\left(\|\boldsymbol{u}_{(0)}^{n+1} - \boldsymbol{u}_{(0)}^{n}\|_{W^{1,\infty}} + \|\boldsymbol{u}_{(0)}^{n} - \boldsymbol{u}_{R,(0)}^{n}\|_{W^{1,\infty}}\right)$$

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$$\rho^0 = \mathrm{const} + O(\varepsilon^2), \qquad p^0 = \mathrm{const} + O(\varepsilon^2), \qquad \nabla\cdot \boldsymbol{u}^0 = O(\varepsilon^2).$$

I.e. $\rho^0_{(1)} = p^0_{(1)} = \nabla\cdot \boldsymbol{u}^0_{(1)} = 0$.

Theorem 2

Let the assumptions of Theorem 1 hold. Let the initial data be well prepared and let $\rho^n_{R,(1)} = 0$ for all n. Then $\rho^n_{(1)} = p^n_{(1)} = \nabla\cdot \boldsymbol{u}^n_{(1)} = 0$ for all n.

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Existence of the Hilbert expansion

$$\frac{\boldsymbol{w}^{n+1}-\boldsymbol{w}^n}{\Delta t}+\nabla\cdot\Big(\boldsymbol{f}(\boldsymbol{w}^n)+\boldsymbol{f}'(\boldsymbol{w}_R^n)(\boldsymbol{w}^{n+1}-\boldsymbol{w}^n)\Big)=0.$$

- It is not clear *a priori* that the Hilbert expansion at t^{n+1} exists!

Theorem 3

Let $\Omega=[-\pi,\pi]$, periodic BCs, let all quantities be sufficiently smooth. Let $\boldsymbol{w}_R$ be constant in space. Let $\gamma\geq 1$. Let $\boldsymbol{w}^n,\boldsymbol{w}_R$ possess a Hilbert expansion. Then $\boldsymbol{w}^{n+1}$ has a Hilbert expansion, i.e.

$$\boldsymbol{w}^{n+1}=\boldsymbol{w}_{(0)}^{n+1}+\varepsilon\boldsymbol{w}_{(1)}^{n+1}+\varepsilon^2\boldsymbol{w}_{(2)}^{n+1}+\dots$$

Proof:

- "Gaussian elimination" on the ODE level.
- 3rd order ODE for "linearized pressure" p_L.
- Solve using Fourier analysis.
- Hilbert expansion for p_L = Fourier series.
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