Pulsed-beam wavelets and the wave equation

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Abstract

Pulsed-beam wavelets [1] are a family of localized solutions of the wave equation in Minkowski space $\mathbb{R}^{3,1}$. They are parameterized by points $x + iy$ in the future and past tube domains of complex Minkowski space (*ie,* with y in the future or past cones). The entire family is obtained from a single member by scaling and Poincaré transformations, and the action extends to the conformal group $SU(2, 2)$. Various subfamilies form *frames,* giving representations of general solutions as superpositions of pulsed-beam wavelets. These representations are intermediate between the extremes of localization by Green functions and delocalization by Fourier integrals. The event x specifies the wavelet's *point and time of emission or absorption,* while y (generalizing 'scale' when $n = 1$) gives its *spacetime extension* as represented by the direction of propagation, beam collimation, and pulse duration. These wavelets have *no sidelobes* and can be focused as sharply as desired by letting y approach the light cone, becoming singular on the ray $y\,\mathbb{R}^+$ for lightlike y. Doppler effects are represented by the action of the Lorentz group. For these reasons, Pulsed-beam wavelets and their electromagnetic counterparts have been proposed as a natural basis for sonar, radar and communications [2–4]. However, to implement these applications it is necessary to *realize* the wavelets by constructing antennas that simulate their sources. A key concept in the construction is the *complex distance* in $\mathbb{C}^n$ and its associated potential theory. This interpolates smoothly from the Euclidean to the Minkowskian regime, generalizing an old theorem by Garabedian [5]. It will be shown that *nonsingular (Huygens) sources* required to radiate and absorb the wavelets can be built by combining two branch cuts associated with the complex distance [6,7]. Interestingly, Sommerfeld in his 1895 *Habilitationsschrift* [8] developed the first correct theory of diffraction by regarding a diffracting screen as a branch cut in a "Riemannian double-space" covering $\mathbb{R}^3$. This suggests the existence of a general theory where branch cuts represent various material properties, including those associated with diffraction and radiation.

References

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