PŘÍKLAD K VĚTĚ 1.1: $f(x,y) = x^2+y^2, \quad (x,y) \in \mathbb{R}^2$

OTÁZKA: $f([0,1]^2) = ?$

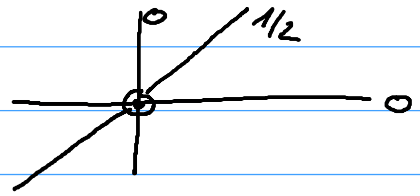
ŘEŠENÍ: $\forall (x,y) \in [0,1]^2: \quad f(0,0) = 0 \leq f(x,y) \leq f(1,1) = 2.$

$$\overset{V1.1}{\Longrightarrow} \quad [0,2] \subseteq f([0,1]^2) \quad (\subseteq [0,2] \text{ platí také – viz})$$

Celkem: $f([0,1]^2) = [0,2].$

POZN: ve větě 1.2 nestačí předpokládat, že $\frac{\partial f}{\partial x_i}$ existují!

PŘ. např.:



$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \dots\ (x,y) \neq (0,0) \\ 0 & \dots\ (x,y) = (0,0) \end{cases}$$

Pak
$$\frac{\partial f}{\partial x}(0,0) = (x \longmapsto 0)'(0) = 0$$
$$\frac{\partial f}{\partial y}(0,0) = (y \longmapsto 0)'(0) = 0$$

ale $f(t,t) = \dfrac{t^2}{2t^2} = \dfrac{1}{2}, \quad t \in \mathbb{R}\setminus\{0\}$

$\Rightarrow f$ není spojitá v $(0,0)$ protože
$$\lim_{t \to 0} f(t,t) = \frac{1}{2} \neq f(0,0).$$

PŘ $f(x,y) = x^2 y + y^3 x^5$. Pak

$$\frac{\partial^2 f}{\partial x\, \partial y}(x,y) = \frac{\partial}{\partial y}\left(2xy + 5y^3x^4\right) = 2x + 15y^2x^4$$

$$\frac{\partial^2 f}{\partial x\, \partial x}(x,y) = 2y + 20y^3x^3$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(x,y) = \frac{\partial}{\partial x}\left(x^2 + 3y^2x^5\right) = 2x + 15y^2x^5$$

$$\frac{\partial f}{\partial y\,\partial y}(x,y) = 6yx^5.$$

Speciálně, $f \in \mathcal{C}^2(\mathbb{R}^2)$.

PŘ

$$f(x,y) = y\sqrt{x}$$

Pak $f \in \mathcal{C}^2(\{(x,y);\ x>0\})$

PŘ FCE, KTERÁ JE $\mathcal{C}^2$ ALE NENÍ $\mathcal{C}^1$:

$$f(x) := \operatorname{sgn}(x)\cdot\frac{x^2}{2}$$

$$(\text{pak } f'(x) = |x|)$$

Pak $f \in \mathcal{C}^1(\mathbb{R}) \setminus \mathcal{C}^2(\mathbb{R})$

VĚTA O IMPLICITNÍ FCI

PŘ $x^2 + 2xy^2 + y^4 - y^5 = 0.$

a) Těmto vztahem je definována fce $y(x)$!

Polož $F(x,y) = x^2 + 2xy^2 + y^4 - y^5$, $x_0 = 0$, $y_0 = 1$. Pak

- $F \in \mathcal{C}^k(\mathbb{R}^2)$ (pro každé $k \in \mathbb{N}$)
- $F(0,1) = 0 + 0 + 1 - 1 = 0$
- $\frac{\partial F}{\partial y}(0,1) = \left(4xy + 4y^3 - 5y^4\right)(0,1) = 4 - 5 = -1 \neq 0$

VOIF

$\Rightarrow \exists \varepsilon > 0$ $\exists \delta > 0$ $\forall x \in (-\varepsilon, \varepsilon)$ $\exists!\, y(x) \in (1-\cancel{\varepsilon}\,\delta, 1+\cancel{\varepsilon}\,\delta)$:

$$F(x, y(x)) = 0$$

Navíc, $y \in \mathcal{C}^k((-\varepsilon, \varepsilon))$ (pro každé $k \in \mathbb{N}$)

a $$y'(x) = -\frac{2x + 2y^2}{4xy + 4y^3 - 5y^4}, \quad x \in (-\varepsilon, \varepsilon). \qquad (*)$$

b) Spočítejte $y'(0)$!

1. ZPŮSOB:

$$y'(0) \overset{(*)}{=} -\frac{2\cdot 0 + 2\cdot y^2(0)}{4\cdot 0\cdot y(0) + 4y^3(0) - 5y^4(0)} \overset{y(0)=1}{=} -\frac{2\cdot 1}{4-5} = \underline{\underline{2}}$$

2. ZPŮSOB:

Máme:

$$0 = F(x, y(x)) = x^2 + 2x\,y^2(x) + y^4(x) - y^5(x),\ x\in(-\varepsilon,\varepsilon)$$

Zderivuj LS a PS:

$$(**)\quad 0 = 2x + \left(2\cdot y^2(x) + 2x\cdot 2\,y(x)\cdot y'(x)\right) + 4y^3(x)\cdot y'(x) - 5y^4(x)\cdot y'(x)$$

(pro $x\in(-\varepsilon,\varepsilon)$)

$$\underset{x=0,\ y(x)=y(0)=1}{\Longrightarrow}\quad 0 = 0 + (2+0) + 4y'(0) - 5y'(0)$$

$$-2 = -y'(0) \Rightarrow \underline{\underline{y'(0)=2}}.$$

c) Spočítejte $y''(0)$!

Zderivuj rovnost $(**)$:

$$0 = 2 + 4y(x)\cdot y'(x) + 4\left(y(x)\cdot y'(x) + x\cdot\left(y'(x)\cdot y'(x) + y(x)\cdot y''(x)\right)\right)$$

$$+ 4\left(3y^2(x)\cdot(y'(x))^2 + y^3(x)\cdot y''(x)\right) - 5\left(4y^3(x)(y'(x))^2 + y^4(x)\cdot y''(x)\right)$$

(pro $x\in(-\varepsilon,\varepsilon)$)

$$\underset{x=0,\ y(0)=1,\ y'(0)=2}{\Longrightarrow}\quad 0 = 2 + 4.2 + 4\left(2 + 0\cdot(\dots)\right) + 4\left(3\cdot 2^2 + y''(0)\right) - 5\left(4\cdot 2^2 + y''(0)\right)$$

$$\Rightarrow\quad 0 = 18 + 4\cdot(12 + y''(0)) - 5(16 + y''(0))$$

$$0 = 18 + 48 - 80 - y''(0)$$

$$y''(0) = 66 - 80 = \underline{\underline{-14}}.$$

c) Je y na okolí 0 konkávní / konvexní?

Víme: $y''(0) = -14$, $y\in\mathcal{C}^2((-\varepsilon,\varepsilon))$

$\Rightarrow \exists\, \tilde{\varepsilon} > 0 : \quad \underline{y''(x) < 0} \quad$ pro $x \in (-\tilde{\varepsilon}, \tilde{\varepsilon})$

DOKONCE

$y''(x) \in (-13, -5)$

$\Rightarrow$ y je konkávní na $(-\tilde{\varepsilon}, \tilde{\varepsilon})$.

<u>Př</u>

$$e^{xy^2-1} + \log \frac{x}{y} = 1 \quad , \quad M = [1,1]$$

<u>(i) Jsou splněny předpoklady VOIF :</u>

$$F(x,y) = e^{xy^2-1} + \log \frac{x}{y} - 1$$

$$G = \{ (x,y) \in \mathbb{R}^2 ;\ x > 0 ,\ y > 0 \}$$

- Pak $F \in \mathcal{C}^k(G)$ pro $k \in \mathbb{N}$

- $F(1,1) = e^{1-1} + \log 1 - 1 = 0$ ✓

- $\frac{\partial F}{\partial y}(1,1) = \left(e^{xy^2-1} \cdot (2xy) + \frac{y}{x} \cdot \left(\frac{-x}{y^2} \right) \right)(1,1)$

$$= e^0 \cdot 2 + 1 \cdot (-1) = 2 - 1 = 1 \neq 0 \ ✓$$

VOIF

$\Rightarrow \exists\, \varepsilon > 0$ $\exists\, \delta > 0$ $\forall x \in (1-\varepsilon, 1+\varepsilon)$ $\exists!\, y(x) \in (1-\delta, 1+\delta)$:

$F(x, y(x)) = 0$

Navíc $\cdot$ $y \in \mathcal{C}^k((1-\varepsilon, 1+\varepsilon))$ pro $k \in \mathbb{N}$

$\cdot$ $y(1) = 1$

(ii) Výpočet $g'(1)$

Máme $\forall x \in (1-\varepsilon, 1+\varepsilon)$:

$$e^{x g^2(x) - 1} + \log \frac{x}{g(x)} = 1$$

zderivujme rovnost, dostaneme pro $x \in (1-\varepsilon, 1+\varepsilon)$:

$$(*) \quad e^{x g^2(x) - 1} \cdot \left(g^2(x) + x \cdot 2\, g(x) \cdot g'(x) \right) + \frac{\cancel{g(x)}}{x} \cdot \frac{g(x) - x \cdot g'(x)}{g^2(x)} = 0$$

$\Rightarrow$

DOSAĎ $x=1,\ g(x)=1$

$$e^{0} \cdot \left(1 + 2 g'(1)\right) + \frac{1 - g'(1)}{1} = 0$$

$$2 + g'(1) = 0 \Rightarrow \underline{\underline{g'(1) = -2}}.$$

(iii) Výpočet $g''(1)$

Zderivujme obě strany rovnosti $(*)$
pro $x \in (1-\varepsilon, 1+\varepsilon)$ dostaneme

$$e^{x g^2(x) - 1} \cdot \left(g^2(x) + 2x\, g(x)\, g'(x) \right)^2 + e^{x g^2(x) - 1} \cdot \Big(2 g(x) g'(x) + 2 g(x) g'(x) + 2x \big(g'(x) g'(x) + g(x) g''(x) \big) \Big)$$

$$+ \frac{\left(\cancel{g'(x)} - \cancel{g'(x)} - x g''(x)\right) x\, g(x) - (g(x) -}{(x\, g(x))^2} = 0$$

$\Rightarrow$

DOSAĎ $x=1,\ g(x)=1,\ g'(x) = -2$

$$e^{0} \cdot \left(1 + 2 \cdot (-2)\right)^2 + e^{0} \cdot \Big(2 \cdot (-2) + 2 \cdot (-2) + 2\left(4 + g''(1)\right) \Big)$$

$$+ \left\{ - g''(1) - (1 - (-2))(1 + (-2)) \right\} = 0$$

$$9 + \cancel{(-4)} + \cancel{(-4)} + \cancel{8} + g''(1) + 3$$

Př

$$\log(x+y^3) + \exp(x+2y) = 1, \quad M = [2,-1]$$

(i) Aplikace VOIF:

$$F(x,y) = \log(x+y^3) + \exp(x+2y) - 1$$

$$G = \{(x,y) \in \mathbb{R}^2 ;\ x + y^3 > 0\}$$

Pak

- $F \in \mathcal{C}^2(G)$
- $F(2,-1) = \log(2-1) + \exp(2-2) - 1 = 0$ ✓
- $\dfrac{\partial F}{\partial y}(2,-1) = \left(\dfrac{1}{x+y^3} \cdot 3y^2 + \exp(x+2y) \cdot 2\right)(2,-1)$

$$= \frac{1}{2-1} \cdot 3 \cdot 1 + \exp(2-2) \cdot 2 = 5 \neq 0 \ ✓$$

$\overset{\text{VOIF}}{\Longrightarrow}$ $\exists \varepsilon > 0$ $\forall x \in (2-\varepsilon, 2+\varepsilon)$ $\exists!\ y(x) \in (-1-\delta, -1+\delta)$:
$\exists \delta > 0$
$F(x, y(x)) = 0$. Navíc, $y \in \mathcal{C}^2((2-\varepsilon, 2+\varepsilon))$, $y(2) = -1$

(ii) Výpočet $y'(2)$:

Máme pro $x \in (2-\varepsilon, 2+\varepsilon)$:

$$\log(x + y^3(x)) + \exp(x + 2y(x)) = 1$$

Zderivujme LS a PS, pro $x \in (2-\varepsilon, 2+\varepsilon)$ dostáváme:

$$(*) \quad \frac{1}{x+y^3(x)} \cdot (1 + 3y^2(x) \cdot y'(x)) + \exp(x+2y(x)) \cdot (1 + 2y'(x)) = 0$$

$\Longrightarrow$ (dosaď $x=2$, $y(x) = -1$)

$$\underbrace{\frac{1}{2-1}}_{=1}(1 + 3y'(2)) + \underbrace{\exp(2-2)}_{=1} \cdot (1 + 2y'(2)) = 0$$

$$2 + 5y'(2) = 0$$

$$\underline{\underline{y'(2) = -2/5}}.$$

Výpočet $g''(2)$:

Zderivujme v (*) LS a PS pro $x \in (2-\varepsilon, 2+\varepsilon)$:

$$-\frac{1}{(x+g^3(x))^2}\cdot\left(1+3g^2(x)\cdot g'(x)\right)^2+\frac{1}{x+g^3(x)}\cdot\left(6g(x)\cdot g'(x)\cdot g'(x)\right.$$
$$\left.+3g^2(x)\cdot g''(x)\right)$$
$$+\exp(x+2g(x))\cdot(1+2g'(x))^2+\exp(x+2g(x))\cdot(2g''(x))=0$$

$\Longrightarrow$

DOSAĎ $x=2$, $g(x)=-1$, $g'(x)=-2/5$

$$-\frac{1}{(2-1)^3}\cdot\left(1+3\cdot(-2/5)\right)^2+1\cdot\left(6\cdot(-1)\cdot(-2/5)^2+3g''(2)\right)$$
$$+\exp(0)\cdot(1-4/5)^2+1\cdot(2g''(2))=0$$

$$-1\cdot\cancel{\frac{1}{25}}+1\cdot\left(-\frac{24}{25}+3g''(2)\right)+\cancel{\frac{1}{25}}+2g''(2)=0$$

$$5g''(2)=\frac{24}{25}, \quad \text{tj.} \quad \underline{\underline{g''(2)=\frac{24}{125}}}.$$

DŮKAZ ČÁSTI VĚTY 1.3 (VOIF):

DŮKAZ EXISTENCE $\varepsilon > 0$:

BÚNO: $\frac{\partial F}{\partial y}(x_0, y_0) > 0$ (přechodem k $-F$)

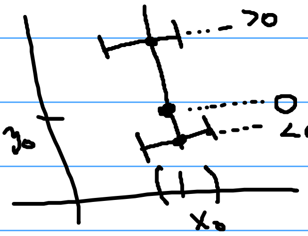
Protože $F \in \mathcal{C}^1(G) \Rightarrow \frac{\partial F}{\partial y}$ je spojitá v (x_0, y_0)

$\Rightarrow \exists\, \eta_0 > 0 : \frac{\partial F}{\partial y}(x,y) > 0$ pro $(x,y) \in B\big((x_0,y_0), \eta_0\big) \subset G$

Položme $\eta = \frac{\eta_0}{\sqrt{2}}$. Pak

$$(x_0 - \eta, x_0 + \eta) \times (y_0 - \eta, y_0 + \eta) \subseteq B\big((x_0, y_0), \eta_0\big)$$

$$\Big[\ \rho\big((x,y),(x_0,y_0)\big) \underset{\substack{|x-x_0|<\eta \\ |y-y_0|<\eta}}{=} \sqrt{(x-x_0)^2 + (y-y_0)^2} < \sqrt{\eta^2+\eta^2} = \sqrt{2}\cdot\eta = \eta_0\ \Big]$$

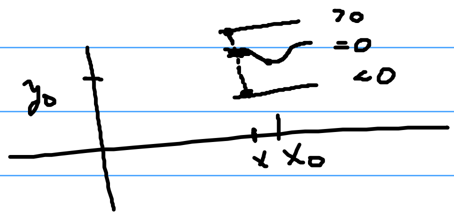


definujme $\psi_0(t) := F(x_0, t)$, $t \in [y_0 - \eta, y_0 + \eta]$

Pak $\psi_0'(t) > 0$, tedy ψ_0 roste na $(y_0-\eta, y_0+\eta)$

$\psi_0(y_0) = 0$, tedy $\psi_0(y_0 - \eta) < 0$
$\psi_0(y_0 + \eta) > 0$

Protože F je spojitá na G, existují $\theta_1, \theta_2 > 0$:



Pak $\psi_x(y_0 - \eta) < 0$, $\psi_x(y_0 + \eta) > 0$

ψ_x je spojitá

$\Rightarrow \exists\, y(x) \in (y_0 - \eta, y_0 + \eta) : \psi_x(y(x)) = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad F(x, y(x))$

$\psi_x' > 0 \Rightarrow \psi_x$ roste $\Rightarrow$ takové $y(x)$ existuje pouze jedno.

☒

konec důkazu existence $\varepsilon > 0$.

Fix $j = 1, \dots, n$.

DK VĚTY 1.4: Ať $g(\lambda) := f(\vec{a} + \lambda \cdot \vec{e_j})$, $\lambda \in (-\varepsilon, \varepsilon)$

(kde $\varepsilon > 0$ je takové, že g je definovaná!)

Pak g má v 0 lok. extrém (protože $\vec{a}$ je lok. extrém f)

$\underset{\text{1. semestr}}{\Longrightarrow}$ $g'(0) = 0$ nebo $g'(0)$ neex. vlastní

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\frac{\partial f}{\partial x_j}(\vec{a}) = 0 \qquad \frac{\partial f}{\partial x_j}(\vec{a}) \text{ neex.}$$

■

KOMENTÁŘE K VĚTĚ 1.5:

(a) $f(x,y) = -x^2 - y^2$

Pak $\nabla f(x,y) = (-2x, -2y) = (0,0)$

$\iff x = 0 \ \& \ y = 0$

Dle 1.4, jediný kandidát na lok. extrém je bod $(0,0)$.

Zároveň

$$\nabla^2 f(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

„$ab = 4 > c^2 = 0$" ... matice je negativně definitní

$\overset{V1.5}{\Longrightarrow}$ v bodě $(0,0)$ je ostré lok. max.

(b) $f(x,y) = x^2 + y^2$

Pak $\nabla f(x,y) = (2x, 2y) = \vec{0}$

$\iff (x,y) = (0,0)$

$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ je pozitivně definitní

$\overset{V1.5}{\Longrightarrow}$ v bodě $(0,0)$ je ostré lok. min.

(c) $f(x,y) = x^2 - y^2$

Pak $\nabla f(x,y) = \vec{0} \iff (x,y) = \vec{0}$

$$\nabla^2 f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{je indefinitní}$$

$$\overset{V1.5}{\Longrightarrow} \quad \text{v bodě } (0,0) \text{ není lok. extrém}$$

POZN:

Pro $f(x,y) = \pm x^4 \pm y^4$ .... $\nabla f(x,y) = \vec{0} \Leftrightarrow (x,y) = \vec{0}$

$$\nabla^2 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ale pro $f(x,y) = x^4 + y^4$ máme lok. min.;

$f(x,y) = -x^4 - y^4$ — " — max;

$f(x,y) = x^4 - y^2$ nemáme lok. extrém

POZN k V 1.6: Je třeba $\underline{f \in \mathcal{C}^2(G)}$ (bez tohoto předpokladu věta neplatí)

PŘ.:

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & \dots\dots (x,y) \neq (0,0) \\ 0 & \dots (x,y) = (0,0) \end{cases}$$

Pak pro $(x,y) \neq (0,0)$ máme

$$\frac{\partial f}{\partial x}(x,y) = y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{2x(x^2+y^2)-(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= y\left(\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^2}{(x^2+y^2)^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y) \underset{\text{ANALOGICKY}}{=} x\left(\frac{x^2-y^2}{x^2+y^2} - \frac{4x^2y^2}{(x^2+y^2)^2}\right)$$

A pro $(x,y) = (0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = (x \mapsto 0)'(0) = 0; \quad \frac{\partial f}{\partial y}(0,0) \overset{\text{ANALOGICKY}}{=} 0$$

Tedy

$$\frac{\partial f}{\partial x\,\partial y}(0,0) = \left(y \longmapsto \frac{\partial f}{\partial x}(0,y)\right)'(0)$$

$$= (y \longmapsto -y)'(0) = \underline{\underline{-1}}$$

ale $\dfrac{\partial f}{\partial y\, \partial x}(0,0) = \left(x \longmapsto \dfrac{\partial f}{\partial y}(x,0)\right)'(0)$

$= (x \longmapsto x)'(0) = \underline{\underline{1}}$

Tedy $\dfrac{\partial f}{\partial x \partial y}(0,0) \neq \dfrac{\partial f}{\partial y \partial x}(0,0)$.

$\boxed{\text{PŘÍKLADY}}$

$$f(x,y) = x^2 + (y-1)^2$$

- $f(n,n) \to \infty$ pro $n \to \infty$, tedy f nemá max.
- $f(x,y) \geq 0 = f(0,1) \Rightarrow$ bod $(0,1)$ je lok. min.

Jiné lok. extrémy nejsou, protože

$$\nabla f(x,y) = (2x,\ 2(y-1)) = \vec{0} \iff (x,y) = (0,1).$$

$$f(x,y) = |x^2+y^2-1|$$

Položme $g(R) := |R-1|$, $R \in [0,\infty)$

$\Rightarrow$ lok. max. v bodě $(0,0)$; lok. min. v bodech

$A := \{(x,y);\ x^2+y^2 = 1\}$

Dk.: Pro $(x,y) \notin A$ máme

$\left.\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \operatorname{sgn}(x^2+y^2-1) \cdot 2x \\ \frac{\partial f}{\partial y}(x,y) &= \text{———}\,\|\,\text{———} \cdot 2y \end{aligned}\right\}\ \nabla f(x,y) = \vec{0} \iff (x,y) = \vec{0}$

$\Rightarrow$ Jediný kandidát v $\{(x,y);\ x^2+y^2 \neq 1\}$ na lok. extrém je bod $(0,0)$.

a v bodě $(0,0)$ je lok max protože

$\nabla^2 f(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ je negativně definitní

V bodech $(x,y) \in A$ je zřejmé globální minimum.

(ale není tam ostatní lok. min.) ⊗

$$f(x,y) = e^{x^2-y}(5-2x+y)$$

⌈ Máme

$$\nabla f(x,y) = \left(e^{x^2-y}\left(2x(5-2x+y) + (-2)\right),\ e^{x^2-y}\left(-(5-2x+y)+1\right)\right)$$

$$= \left(e^{x^2-y}\left(-4x^2+10x+2xy-2\right),\ e^{x^2-y}\left(-4+2x-y\right)\right)$$

$$= (0,0)$$

$$\iff -2x^2+5x+xy-1=0 \ \& \ -y-4+2x=0$$

$$\iff y = 2x-4 \ \& \ -2x^2+5x+x(2x-4)-1=0$$

tj. $x=1$

$$\iff x=1 \ \& \ y=-2$$

Tj. stac. bod je jen $(x,y) = (1,-2)$.

($\overset{V1.4.}{\Longrightarrow}$ Pokud (x,y) je lok. extrém, pak $(x,y) = (1,-2)$)

$$\nabla^2 f(1,-2) \overset{V1.6}{=} \begin{pmatrix} e^{x^2-y}\left(2x(-4x^2+2xy+10x-2) + (-8x+2y+10)\right) & \cdots\cdots \\ e^{x^2-y}\left(-(-4x^2+2xy+10x-2)+2x\right) & e^{x^2-y}\left(-(-4+2x-y) + (-1)\right) \end{pmatrix}_{(1,-2)}$$

$$= \begin{pmatrix} e^3(2\cdot 0 + (-8-4+10)) & 2e^3 \\ e^3(-0+2) & e^3(-0-1) \end{pmatrix} = \begin{pmatrix} -2e^3 & 2e^3 \\ 2e^3 & -e^3 \end{pmatrix}$$

$\Longrightarrow$ $\nabla^2 f(1,-2)$ je indefinitní

„$ab = 2e^6 < 4e^6 = c$"

$\Longrightarrow f$ nemá lok. extrémy ⌋

$$f(x,y) = xy\sqrt{1-x^2-y^2}\ ;\quad (x,y)\in G := \{(x,y);\ x^2+y^2<1\}$$

$\ulcorner\ \forall (x,y)\in G:$

$$\nabla f(x,y) = \Bigg(y\underbrace{\left(\sqrt{1-x^2-y^2} + x\,\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2-y^2}}\right)}_{=\frac{1-2x^2-y^2}{\sqrt{1-x^2-y^2}}},\quad x\,\frac{1-2y^2-x^2}{\sqrt{1-x^2-y^2}}\Bigg)$$

$$= (0,0)$$

$$\Longleftrightarrow \left(y=0 \text{ nebo } 2x^2 = 1-y^2\right)\ \&\ \left(x=0 \text{ nebo } 2y^2 = 1-x^2\right)$$

$$\Longleftrightarrow \big((x,y)=(0,0)\big) \text{ nebo } \big((x,y) = \overbrace{(\pm 1, 0)}^{\notin G}\big) \text{ nebo } \big((x,y) = \overbrace{(0,\pm 1)}^{\notin G}\big)$$

$$\text{nebo } \left(y^2 = 1-2x^2\ \&\ 1-x^2 = 2y^2 = 2(1-2x^2)\right)$$

$$\left[\ 3x^2 = 1,\ \text{tj. } |x| = \frac{1}{\sqrt{3}},\quad \text{a tedy } |y| = \frac{1}{\sqrt{3}}\ \right]$$

$$\Longleftrightarrow \big((x,y) = (0,0)\big) \text{ nebo } \left(|x| = |y| = \frac{1}{\sqrt{3}}\right)$$

<u>Zbývá</u>: SPOČÍTAT V TĚCH TO BODECH $\nabla^2 f$:

$$\nabla^2 f(x,y) \overset{\forall\in G}{=} \begin{pmatrix} y\,\dfrac{-4x\sqrt{1-x^2-y^2} - (1-2x^2-y^2)\frac{-2x}{2\sqrt{1-x^2-y^2}}}{(1-x^2-y^2)} & \cdots\cdots \\[2ex] \dfrac{1-2x^2-y^2}{\sqrt{1-x^2-y^2}} + y\cdot\dfrac{-2y\sqrt{1-x^2-y^2} - (1-2x^2-y^2)\frac{-y}{\sqrt{1-x^2-y^2}}}{1-x^2-y^2} & \end{pmatrix}$$

analogicky, jen místo y píši x a naopak

$$= \ldots\ldots =$$

v BODĚ $(0,0)$

$$\nabla^2 f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

v BODECH $|x| = |y| = \frac{1}{\sqrt{3}}$:

$$\nabla^2 f(x,y) = \begin{pmatrix} \operatorname{sgn}(yx)\left(\frac{\sqrt{3}}{3}+\sqrt{3}\right) & -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & -\operatorname{sgn}(yx)\,\frac{4\sqrt{3}}{3} \end{pmatrix}$$

V1.5

$\Rightarrow$ v bodě $(0,0)$ sedlový bod

ostré lok. max. v bodech $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ a $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

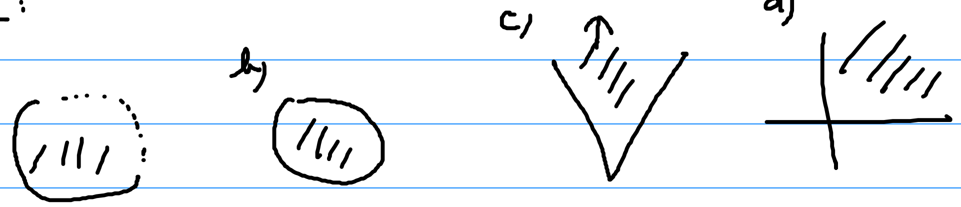
—"— min. —"— $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ a $\left(-\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

EXTRÉMY FCÍ VÍCE PROMĚNNÝCH II

KOMPAKTNOST:

Př

a) b) c) d)



e)

jsou kompaktní?

f)

Řešení:

a) NE (není uz.)

b) ANO

c) NE (není om.)

d) —"—

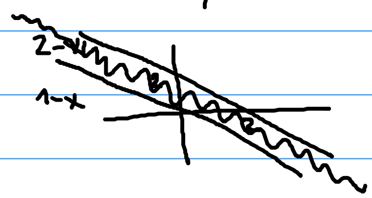
e) ANO

f) ANO

Př

a) $\{(x,y);\ x+y \leq 2\}$ ... není kpt. proto není om.

b) $\{(x,y);\ 1 \leq x+y \leq 2\}$ ... není kpt. proto není om.

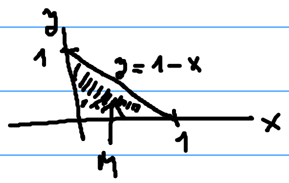


c) $\{(x,y);\ x^2+y^2 \leq 100\}$ ... je KM

Př

$f(x,y) = x-2y-3 \quad ; \quad M = \{x \in [0,1],\ y \in [0,1],\ x+y \leq 1\}$

ob.:



M je kompaktní $\Rightarrow$ f nabývá max a min. na M

1. krok: $G := \{x \in (0,1),\ y \in (0,1),\ x+y < 1\}$ množina je ot.

Pokud $\vec{a} \in G$ je lok. extrém f, pak

$$\nabla f(\vec{a}) = \vec{0}$$

Ale $\nabla f(x,y) = (1,-2)$; $(x,y) \in G$

$\Rightarrow$ v G fce f nemá lok. extrém.

<u>2. krok</u>:

$H_1 = \{(0,y);\ y \in [0,1]\}$

$H_2 = \{(x,0);\ x \in [0,1]\}$

$H_3 = \{(x,y);\ x+y=1,\ x \in [0,1]\}$

Pak $M \setminus G = H_1 \cup H_2 \cup H_3$

Vyšetříme extrémy na H_i, $i \in \{1,2,3\}$:

<u>H_1</u>: Pro $(x,y) \in H_1$ je $f(x,y) = f(0,y) = -2y-3$, $y \in [0,1]$

$\Rightarrow$ ~~min~~ klesá (jakožto fce závislá na y)

$$\min_{y\in[0,1]} f(0,y) = f(0,1) = -5 \ ; \ \max_{(x,y)\in H_1} f(x,y) = f(0,0) = -3$$

<u>H_2</u>: Pro $(x,y) \in H_2$ je $f(x,y) = f(x,0) = x-3$, $x \in [0,1]$

$\Rightarrow$ roste (jakožto fce závislá na $\underline{x}$)

$$\min_{(x,y)\in H_2} f(x,y) = f(0,0) = -3 \ ; \ \max_{(x,y)\in H_2} f(x,y) = f(1,0) = -2$$

<u>H_3</u>: Pro $(x,y) \in H_3$ je $f(x,y) = f(x, 1-x)$

$$= x - 2(1-x) - 3 = 3x - 5, \ x \in [0,1]$$

$\Rightarrow$ roste (jakožto fce závislá na $\underline{x}$)

$$\min_{(x,y)\in H_3} f(x,y) = f(0,1) = -5 \ ; \ \max_{(x,y)\in H_3} f(x,y) = f(1,0) = -2$$

<u>Celkem</u>:

$$\max_{(x,y)\in M} f = f(1,0) = -2 \ ; \ \min_{(x,y)\in M} f(x,y) = f(0,1) = -5.$$

PŘ

$$f(x,y,z) = (x+y)^2 + (x-y)^2 + z, \quad M = [-1,1]^3$$

- M je kompakt $\Rightarrow$ f nabývá max. a min. (protože je spoj.)

1. KROK: $G := (-1,1)^3$ ... je ot. množina

$$\nabla f(x,y,z) = \big(2(x+y)+2(x-y),\ 2(x+y)-2(x-y),\ 1\big) \neq \vec{0}$$

$\Rightarrow$ f nemá v G lok. extrém.

2. KROK: Vyšetřujeme extrémy na hranici

a) AŤ $x = 1$; ~~f~~ Pak $f(x,y,z) = f(1,y,z) = (1+y)^2 + (1-y)^2 + z$, $(y,z) \in [-1,1]^2$

Položme $g(y,z) := f(1,y,z)$, $(y,z) \in [-1,1]^2$

a zjistěme extrémy g na $[-1,1]^2$ ... to je kompakt, tj. $g(y,z)$ nabývá max. a min.

- $\nabla g(y,z) = \big(2(1+y) - 2(1-y),\ 1\big) \neq \vec{0}$

  $\Rightarrow g$ nemá lok. extrém v $(-1,1)^2$

- POKUD $y = 1$:

  $g(y,z) = g(1,z) = 4+z$ .. ROSTOUCÍ (jakožto fce závislá na z)

  $\Rightarrow$ na množině $\{(y,z) \in [-1,1]^2;\ y = 1\} =: \tilde{H}_1$

  máme
  $$\min_{\tilde{H}_1} g = g(1,-1) = 3; \quad \max_{\tilde{H}_1} g = g(1,1) = 5$$

- POKUD $y = -1$:

  $g(y,z) = g(-1,z) = 4+z$ ... tady situace stejná jako

  $\Rightarrow$ na $\tilde{H}_2 = \{(y,z) \in [-1,1]^2;\ y = -1\}$

  je
  $$\min_{\tilde{H}_2} g = g(-1,-1) = 3; \quad \max_{\tilde{H}_2} g = g(1,1) = 5$$

• POKUD $z=1$:

$$g(y,z) = (1+y)^2 + (1-y)^2 + 1 = 3 + 2y^2 \,,\; y \in [-1,1]$$

$\Rightarrow$ extrémy pro $y \in \{-1, 0, 1\}$

• POKUD $z=-1$:

$$g(y,z) = 1+2y^2 \,,\; y \in [-1,1]$$

$\Rightarrow$ extrémy pro $y \in \{-1, 0, 1\}$

Celkem: extrémy na $\{(x,y,z) \in [-1,1]^3 ;\ x=1\}$ fce f jsou v množině

$$\{ (1, \pm 1, \pm 1) \,;\, (1, 0, \pm 1) \}$$

4 VARIANTY ZNAMÉNEK

b) $x=-1$: Máme $f(-1, y, z) = f(1, y, z)$

$\Rightarrow$ dle a) extrémy jsou v množině

$$\{(-1, \pm 1, \pm 1) \,;\, (-1, 0, \pm 1)\}$$

c) Ať $x \neq \{\pm 1\}$, $y=1$:

$f(x,1,z) = f(1,x,z)$ .... dle a) extrémy v množině

$$\{(0, 1, \pm 1)\}$$

d) Ať $x \notin \{\pm 1\}$, $y=-1$: $f(x,-1,z) = f(-1,x,z)$ ... dle b) extrémy v množině $\{(0,-1,\pm 1)\}$

e) Ať $x,y \notin \{\pm 1\}$, $z = \pm 1$:

$$f(x,y,\pm 1) = (x+y)^2 + (x-y)^2 \pm 1 =: h(x,y) \,,\; (x,y) \in (-1,1)^2$$

Máme
$$\nabla h(x,y) = \Big(\overbrace{2(x+y) + 2(x-y)}^{2(2x)} \,,\, \overbrace{2(x+y) - 2(x-y)}^{4y} \Big) = \vec{0}$$

$$\iff (x,y) = (0,0)$$

Tedy lok. extrémy h na $(-1,1)^2$ jsou pouze $\{(0,0)\}$

$\Rightarrow$ v případě e) může mít fce extrémy pouze v $\{(0,0,\pm 1)\}$

Celkem: $\max\limits_{M} f = \max \{ f(\pm 1, \pm 1, \pm 1), f(\pm 1, 0, \pm 1), f(0, \pm 1, \pm 1), f(0,0,\pm 1) \}$

$\min\limits_{M} f = \min \{ \text{—— " ——} \}$

DOSADÍ SE .... A UČINÍ ZÁVĚR ...

Př

$$f(x,y) = \operatorname{arctg} x + \operatorname{arctg} y \; ; \quad M = \{ x^2+y^2 \leq 1, \, x \geq 0, \, y \geq 0 \}$$

• M je kompakt, f spojitá $\Rightarrow$ f nabývá na M max. a min.

1. krok: $G := \{ x^2+y^2 < 1, \, x > 0, \, y > 0 \}$ je ot.

$$\nabla f(x,y) = \left(\frac{1}{1+x^2}, \frac{1}{1+y^2} \right) \neq \vec{o}$$

$\Rightarrow$ f nemá v G lok. extrém

2. krok: Hranici rozdělím na 3 množiny:

$H_1 = \{ x=0, \, y \geq 0, \, x^2+y^2 \leq 1 \} = \{ x=0, \, y \in [0,1] \}$

$H_2 = \{ x \geq 0, \, y=0, \, x^2+y^2 \leq 1 \} = \{ y=0, \, x \in [0,1] \}$

$H_3 = \{ x>0, y>0, \, x^2+y^2 = 1 \} = \{ \overset{1}{\vee} x>0, y>0, \, y = \sqrt{1-x^2} \}$

$\{ 1 > x > 0, \, y = \sqrt{1-x^2} \}$

H_1: pro $(x,y) \in H_1$ máme

$f(x,y) = f(0,y) = \operatorname{arctg} y$ .... rostoucí fce

$\Rightarrow \min\limits_{H_1} f = f(0,0) = 0 \; ; \quad \max\limits_{H_1} f = f(0,1) = \operatorname{arctg} 1 = \frac{\pi}{4}$

H_2: Ze symetrické role x a y a H_1 máme

$\min\limits_{H_2} f = f(0,0) = 0 \; ; \quad \max\limits_{H_2} f = f(1,0) = \frac{\pi}{4}$

$\underline{H_3}$: Pro $(x,y) \in H_3$ máme

$$f(x,y) = f(x, \sqrt{1-x^2}) = \operatorname{arctg} x + \operatorname{arctg} \sqrt{1-x^2} =: g(x), \quad x \in [0,1)$$

Je třeba zjistit extrémy g na $(0,1)$:

$$g'(x) = \frac{1}{1+x^2} + \frac{1}{1+1-x^2} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$$

$$= \frac{(2-x^2)\sqrt{1-x^2} - x(1+x^2)}{\underbrace{(1+x^2)(2-x^2)\sqrt{1-x^2}}_{>0}} > 0$$

$$\iff (2-x^2)\sqrt{1-x^2} > x(1+x^2) \quad /^2$$

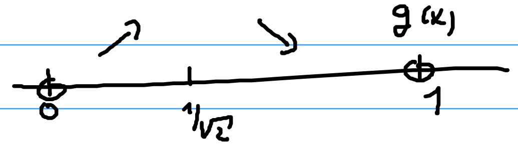
$$\underbrace{(2-x^2)^2(1-x^2)}_{4-8x^2+5x^4-x^6 = (4-4x^2+x^4)(1-x^2)} > \underbrace{x^2(1+x^2)^2}_{x^2+2x^4+x^6}$$

$$\iff 0 > 2x^6 - 3x^4 + 9x^2 - 4 \quad \text{.... polož } z := x^2$$

$$\iff 0 > \underbrace{2z^3 - 3z^2 + 9z - 4}_{2(z-1/2)\underbrace{(z^2 - z + 4)}_{>0}}$$

$$\iff x^2 < 1/2 \iff x < 1/\sqrt{2}$$

$\underline{\text{tj.}}$:



$\Rightarrow$ lok. extrém g je jen v bodě $x = 1/\sqrt{2}$

$\Rightarrow$ na H_3 má f lok. extrém v bodě

$$\left(1/\sqrt{2}, 1/\sqrt{2}\right)$$

$\underline{\text{Celkem}}$:

$$\max_M f = \max\{\overbrace{f(1/\sqrt{2}, 1/\sqrt{2})}^{2\operatorname{arctg}\frac{1}{\sqrt{2}}}, f(0,0), \overbrace{f(0,1)}^{\frac{\pi}{4}}, \overbrace{f(1,0)}^{\pi/4}\} = 2\operatorname{arctg}\frac{1}{\sqrt{2}} = f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

$$\min_M f = \min\{\text{———} \;\; '' \;\; \text{———}\} = 0 = f(0,0).$$

PŘ

$$M = \{(x,y) \in \mathbb{R}^2 ;\ x^2+y^2 = 136\}$$

$$f(x,y) = 5x - 3y$$

Řešení:

• M je uz. + om., tedy bylo $\overset{\text{SPOJITOST } f}{\Longrightarrow}$

f nabývá max a min na M

• dle Lagr. věty o multiplik. (1-8), POKUD $(x_0, y_0) \in M$ je lok. lok. extrém, PAK

[APLIKUJI NA $g(x,y) = x^2+y^2-136$; $G = \mathbb{R}^2$]

BUĎ: $\nabla g(x_0, y_0) = \vec{0}$

$\left(\Leftrightarrow (2x_0, 2y_0) = \vec{0} \Leftrightarrow x_0 = y_0 = 0 \right.$,

ale $(0,0) \notin M$, tedy tato možnost nenastane

nikdy$\left. \right)$

NEBO: $\exists \lambda \in \mathbb{R}: \nabla f(x_0, y_0) + \lambda \nabla g(x_0, y_0) = \vec{0}$, tj.

$$\exists \lambda \in \mathbb{R}: \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2x_0 \\ 2y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To se stane $\Leftrightarrow \left[\begin{array}{l} 5 + \lambda 2 x_0 = 0 \\ -3 + 2\lambda y_0 = 0 \\ x_0^2 + y_0^2 = 136 \end{array} \right]$

$\Leftrightarrow \exists \lambda' \in \mathbb{R} : \lambda' x_0 = -5$ $\quad x_0^2+y_0^2 = 136$

$\lambda' y_0 = 3$

$\Leftrightarrow \exists \lambda' \in \mathbb{R}: \lambda' = -\frac{5}{x_0} = \frac{3}{y_0}$ & $x_0^2+y_0^2 = 136$

$\Leftrightarrow y_0 = -\frac{3x_0}{5} \neq 0$ & $x_0^2 + y_0^2 = 136$

$\Rightarrow x_0^2 + \frac{9}{25} x_0^2 = 136$

$x_0^2 = 136 \cdot \frac{25}{34} = \frac{68 \cdot 25}{17} = 100$

$|x_0| = 10.$

$\iff (x_0, y_0) \in \{(10,-6), (-10, 6)\}$

Tedy $\max\limits_M f = \max\{f(10,-6), f(-10,6)\} = f(10,-6) = \underline{\underline{68}}$

$\min\limits_M f = \min\{ \text{——} \,\,\text{"}\,\, \text{——} \} = f(-10,6) = \underline{\underline{-68}}$

PE

$M = \{x^2+y^2 \le 1, \ x \geq 0, \ y \geq 0\}$, $f(x,y) = \operatorname{arctg} x + \operatorname{arctg} y$

MINULE: • Na M f nabývá max. a min.

• Pokud je extrém na $(M \cap \{x^2+y^2<1\}) \cap (M \cap \{x=0\}) \cap (M \cap \{y=0\})$

Tak je to jeden z bodů $(0,0), (1,0), (0,1)$.

• Vyšetříme extrémy na $H_3 = \{x>0, y>0, \ x^2+y^2=1\}$

Aplikujeme Lagr. multiplik.

$[\ G = \{(x,y) \in \mathbb{R}^2;\ x>0 \ \& \ y>0\};\ g(x,y) = x^2+y^2-1\ ]$

průnik ot. množin $\Rightarrow$ ot. množina

Tedy, pokud $(x,y) \in H_3$ je lok. extrém, pak

BUĎ: $\nabla g(x,y) = \vec{0}$ $\quad (\iff (x,y) = (0,0) \notin H_3$

.... tato možnost tedy nenastane)

NEBO: $\exists \lambda \in \mathbb{R}: \ \nabla f(x,y) + \lambda \nabla g(x,y) = \vec{0}$, tj:

$$\exists \lambda \in \mathbb{R}: \begin{pmatrix} \frac{1}{1+x^2} \\ \frac{1}{1+y^2} \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$x>0, y>0$ protože $(x,y) \in G$

$$\iff \frac{1}{x(1+x^2)} = \frac{1}{y(1+y^2)} \quad (\& \ x^2+y^2=1)$$

$y^2 = 1-x^2$

$$\iff \frac{1}{x(1+x^2)} = \frac{1}{y(2-x^2)}$$

$y = \sqrt{1-x^2}$

$$\iff \sqrt{1-x^2}\,(2-x^2) = x(1+x^2) \quad (\& \ x \in (0,1))$$

$$\iff \quad (1-x^2)(4-4x^2+x^4) = x^2(1+2x^2+x^4)$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ & $x \in (0,1)$

Položme $z = x^2$ ... řešíme rci

$$(1-z)(4-4z+z^2) = z(1+2z+z^2)$$

$\rightsquigarrow$ tj. .... $\quad -2z^3+3z^2-9z+4 = 0$

.... (viz. minulé př. III.z.c.) ... $z = 1/2$

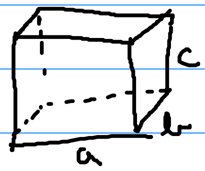
$$\Rightarrow x^2 = 1/2 \underset{\substack{x>0 \\ y>0}}{\Rightarrow} x = \frac{1}{\sqrt{2}}, \quad y = \frac{1}{\sqrt{2}}$$

Celkem: jediný kandidát na lok. extrém na M_3 je bod $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

... a dál se postupuje stejně jako minule ...

APLIKACE – EXTRÉMY FCÍ VÍCE PROMĚNNÝCH

Př. $\quad V = abc$



$S = ab + 2(ac+bc)$

Chci: $\quad \min\limits_{M} f$, kde $f(a,b,c) = ab + 2(ac+bc)$

$$M = \{(a,b,c);\ abc = V,\ a>0,\ b>0,\ c>0\}$$

Pak $\quad c = \frac{V}{ab}, \quad f(a,b,c) = ab + 2\left(\frac{V}{b} + \frac{V}{a}\right)$

$$= ab + 2V\left(\frac{1}{b} + \frac{1}{a}\right) =: g(a,b)$$

Tedy hledám $\min\limits_{(0,\infty)^2} g$.

- Hledáme stacionární body g:

$$\nabla g(a,b) = \left(b + 2V\frac{-1}{a^2},\ a + 2V\frac{-1}{b^2}\right) = (0,0)$$

$$\iff b = \frac{2V}{a^2} \quad \& \quad a = \frac{2V}{b^2}$$

$$\rightarrow a = \frac{2V\, a^4}{(2V)^2} = \frac{a^4}{2V}$$

$$2V = a^3$$

$$\iff a = \sqrt[3]{2V}, \quad b = 2V \cdot (2V)^{-2/3} = \sqrt[3]{2V}$$

Tedy, pokud má g lok. extrém na $(0,\infty)^2$, pak je to v bodě

$$\left(\sqrt[3]{2V}, \sqrt[3]{2V}\right)$$

$\Rightarrow$ POKUD f má extrém v M, pak je to v bodě

$$\left(\sqrt[3]{2V}, \sqrt[3]{2V}, \frac{V}{\sqrt[3]{4V^2}}\right) = \left(\sqrt[3]{2V}, \sqrt[3]{2V}, \sqrt[3]{\frac{V}{4}}\right)$$

Řešení: ROZMĚRY JSOU $\sqrt[3]{2V}, \sqrt[3]{2V}, \sqrt[3]{\frac{V}{4}}$.

PŘ Dáno $a>0$.

CHCEME $\min_M f$, kde $f(x,y) = x^2+y^2$

$M = \{(x,y)\in\mathbb{R}^2;\ x+y=a\}$

1. METODA LAGR. MULTIPLIKÁTORY:

APLIKUJI VĚTU O L. MULTIPLIK. $[G=\mathbb{R}^2,\ g(x,y) = x+y-a]$

POKUD ex. $\min_M f$ a (x,y) je bodem lok. extrému, pak

BUĎ: $\nabla g(x,y) = \vec{0} \iff (1,1) = \vec{0}$ ... to se nikdy nestane

$g(x,y) = x+y-a$

NEBO (JINAK): $\nabla f(x,y) + \lambda \nabla g(x,y) = \vec{0}$

$$\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

tj. $x=y$ $(\&\ x+y=a)$

tedy $x=y=a-y$, tj. $\underline{\underline{y=\frac{a}{2}=x}}$.

Tedy, pokud má úloha řešení, pak je jím bod $(x,y) = \left(\frac{a}{2}, \frac{a}{2}\right)$

2. METODA - BEZ MULTIPLIK.

Máme $y = a-x$, tedy $f(x,y) = x^2+(a-x)^2 =: \varphi(x)$

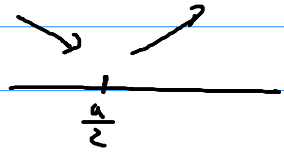
a hledáme $\min_{\mathbb{R}} \varphi$

dále

$$h'(x) = 2x - 2(a-x) = 4x - 2a = 0$$

$$\Longleftrightarrow \quad x = a/2$$

dk.



Tedy $\min\limits_{\mathbb{R}} h = h(a/2)$

$$\Rightarrow \min_M f = f\left(\frac{a}{2}, \frac{a}{2}\right) = \underline{\underline{\frac{a^2}{2}}}.$$

Př.

$$\frac{x^n + y^n}{2} \geq \left(\frac{x+y}{2}\right)^n, \quad n \geq 1, \quad x \geq 0, \quad y \geq 0$$

FIX $c \in \mathbb{R}$.

Hledáme $\min\limits_M f$, kde $f(x,y) = \dfrac{x^n + y^n}{2}$

$$M = \{(x,y) \in (0,\infty)^2;\ x + y = c\}$$

Chci: $\min\limits_M f \geq \left(\frac{c}{2}\right)^n$

Pro $x = 0$ nebo $y = 0$ je nerovnost zřejmá

• Aplikujeme větu o Lagr. multiplik.

$\left[G = \{(x,y) \in \mathbb{R}^2,\ x > 0\ \&\ y > 0\};\ g(x,y) = x + y - c \right]$

průnik ot. množin ⇒ ot. množina

Tedy, pokud $(x,y) \in M$ je bodem lok. extrému f na M, pak

Buď: $\nabla g(x,y) = (1,1) = \vec{0}$ .... nestane se nikdy

nebo: $\exists \lambda \in \mathbb{R}$: $\nabla f(x,y) + \lambda \nabla g(x,y) = \vec{0}$, tj.

$$\exists \lambda \in \mathbb{R}: \quad \frac{1}{2} n \begin{pmatrix} x^{n-1} \\ y^{n-1} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0}$$

tedy $x^{n-1} = y^{n-1} \underset{x>0,\ y>0}{\Longrightarrow} x = y \Rightarrow \underline{\underline{x = y = \frac{c}{2}}}$

⇒ jediný kandidát na lok. extrém je v bodě $\left(\frac{c}{2}, \frac{c}{2}\right)$

Celkem: $\tilde{M} = \{ (x,y) \in \mathbb{R}^2 ;\ x \geq 0,\ y \geq 0,\ x+y = c \}$

je kompakt (PRŮNIK TŘÍ UZ. MNOŽIN $\Rightarrow$ JE UZ. $\subseteq [0,c]^2$, TEDY JE TO OM. MNOŽINA)

$\Rightarrow \min_{\tilde{M}} f$ existuje a nabývá se v jednom z bodů

$$\left(\frac{c}{2}, \frac{c}{2}\right), (0,c), (c,0)$$

ale $f\left(\frac{c}{2}, \frac{c}{2}\right) = \left(\frac{c}{2}\right)^n$

$f(0,c) = \frac{c^n}{2} = f(c,0) \geq \left(\frac{c}{2}\right)^n$

Celkem: $\min_{\tilde{M}} f \geq \left(\frac{c}{2}\right)^n$, což jsme chtěli dokázat.

<u>Důkaz V1.8</u>: Fix $a \in G$.

Zvol $\delta > 0$: $B(b, \delta) \subset H$

Pro $\eta = \frac{\delta}{\sqrt{2}}$: $(b_1 - \eta, b_1 + \eta) \times (b_2 - \eta, b_2 + \eta) \subset H$

$\left[|x - b_1| < \eta \ \&\ |y - b_2| < \eta \Rightarrow \|(x,y) - b\| < \sqrt{2}\eta = \delta \right]$

Ze spojitosti φ_i .. $\exists \tau > 0$: $\varphi_1(a-\tau, a+\tau) \subset (b_1 - \eta, b_1 + \eta)$
$\&\ \varphi_2(a-\tau, a+\tau) \subset (b_2 - \eta, b_2 + \eta)$

Pro $x \in (a-\tau, a+\tau)$ z Lagr. věty najdu

- $\xi_1(x) \in (b_1, \varphi_1(x))$: $f(\varphi_1(x), \varphi_2(x)) - f(b_1, \varphi_2(x)) = \frac{\partial f}{\partial x}(\xi_1(x), \varphi_2(x)) (\varphi_1(x) - b_1)$

- $\xi_2(x) \in (b_2, \varphi_2(x))$: $f(b_1, \varphi_2(x)) - f(b_1, b_2) = \frac{\partial f}{\partial y}(b_1, \xi_2(x)) (\varphi_2(x) - b_2)$

Protože $\lim_{x \to a} \varphi_i(x) = b_i$, máme $\lim_{x \to a} \xi_i(x) = b_i$

dále
$$\lim_{x \to a} \frac{F(x) - F(a)}{x - a} = \lim_{x \to a} \frac{1}{x-a} \Big(f(\varphi_1(x), \varphi_2(x)) - f(b_1, \varphi_2(x)) + f(b_1, \varphi_2(x)) - f(b_1, b_2) \Big)$$

$$= \lim_{x\to a} \frac{1}{x-a}\left(\frac{\partial f}{\partial x}(\xi_1(x),\varphi_2(x))\,(\varphi_1(x)-\varphi(a)) + \frac{\partial f}{\partial y}(b_1,\xi_2(x))\,(\varphi_2(x)-\varphi_2(a))\right)$$

$$= \underbrace{\lim_{x\to a} \frac{\partial f}{\partial x}(\xi_1(x),\varphi_2(a))}_{=\frac{\partial f}{\partial x}(b_1,b_2)} \cdot \underbrace{\lim_{x\to a} \frac{\varphi_1(x)-\varphi_1(a)}{x-a}}_{=\varphi_1'(a)} + \frac{\partial f}{\partial y}(b_1,b_2)\,\varphi_2'(a)$$

$\boxtimes$

<u>DK V 1.9</u>:

Ať neplatí (a), chci: platí (b)

BÚNO $\frac{\partial}{\partial y} g(x_0,y_0) \neq 0$

Ať $\varepsilon > 0$ a f má extrém v (x_0,y_0) na

$$\underbrace{B((x_0,y_0),\varepsilon)}_{=:G'} \cap M$$

$\overset{\text{VoIF}}{\Longrightarrow} \exists \delta>0\ \exists \delta'>0\ \exists \varphi: (x_0-\delta, x_0+\delta) \to (y_0-\delta', y_0+\delta')$:

$g(x,\varphi(x)) = 0$ pro $x \in (x_0-\delta, x_0+\delta)$

& $\varphi \in \mathcal{C}^1(x_0-\delta, x_0+\delta)$ & $\varphi(x_0) = y_0$

Označ $h(x) := f(x,\varphi(x))$ .... pak dle V 1.8 $h \in \mathcal{C}^1(x_0-\delta, x_0+\delta)$

x_0 je extrém h na $(x_0-\delta, x_0+\delta)$

$\ulcorner$ (x_0,y_0) je lok. max f

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta)$:

$h(x) = f(x,\varphi(x)) \leq f(x_0,y_0) = h(x_0)$

... podobně pro min. $\lrcorner$

$\Rightarrow h'(x_0) = 0$.

Zároveň

$$h'(x_0) \overset{V1.8}{=} \frac{\partial f}{\partial x}(x_0,y_0)\cdot 1 + \frac{\partial f}{\partial y}(x_0,y_0)\cdot \varphi'(y_0)$$

$$\overset{\text{VoIF}}{=} \text{—— '' ——} \left(-\frac{\frac{\partial g}{\partial x}(x_0,y_0)}{\frac{\partial g}{\partial y}(x_0,y_0)}\right)$$

Stačí pak položit $\lambda := -\dfrac{\frac{\partial f}{\partial y}(x_0, y_0)}{\frac{\partial f}{\partial x}(x_0, y_0)}$. $\boxtimes$

STEJNOMĚRNÁ KONVERGENCE

$f_n \to f \overset{DEF}{\equiv} \forall x \in E\ \forall \varepsilon > 0\ \exists k\ \forall n \geq k : |f_n(x) - f(x)| < \varepsilon$

$f_n \rightrightarrows f \overset{DEF}{\equiv} \forall \varepsilon > 0\ \exists k\ \forall x \in E\ \forall n \geq k$: —— " ——

FAKT 2.1: $f_n \rightrightarrows f \implies f_n \to f$

Dk:

FIX $x_0 \in E$, $\varepsilon > 0$.

Dle p.p., $\exists k\ \underline{\forall x \in E\ \forall n \geq k : |f_n(x) - f(x)| < \varepsilon}$

ZVOLME tedy $k \in \mathbb{N}$ že platí ?

FIX $n \geq k$. PAK $|f_n(x_0) - f(x_0)| < \varepsilon$ položme $x = x_0$ $\boxtimes$

Tvrzení 2.2: $f_n \rightrightarrows f \iff \sigma_n \to 0$, kde $\sigma_n = \sup_{x \in E} |f_n(x) - f(x)|$

Dk.:

„$\Rightarrow$" FIX $\varepsilon > 0$.

Protože $f_n \rightrightarrows f$, můžeme najít n_0 splňující

$\forall x \in E\ \forall n \geq n_0 : |f_n(x) - f(x)| < \varepsilon$

Pak ale

$\forall n \geq n_0 : \underbrace{\sup_{x \in E} |f_n(x) - f(x)|}_{= \sigma_n} < \varepsilon$

„$\Leftarrow$" FIX $\varepsilon > 0$.

Protože $\sigma_n \to 0$, můžeme najít $n_0 \in \mathbb{N}$ splňující

$\forall n \geq n_0 : \sigma_n < \varepsilon$

POLOŽ $k = n_0$. FIX $x \in E$ a $n \geq k$. Pak

$|f_n(x) - f(x)| \leq \sigma_n < \varepsilon$ $\boxtimes$

PŘ.: $f_n(x) = x^n$, $x \in [0,1]$.

Pak $f_n(x) \to$ 0 ... pro $x<1$, 1 .... pro $x=1$.

Položme $f(x) := \begin{cases} 0 \dots & x<1 \\ 1 \dots & x=1 \end{cases}$

Pak $f_n \to f$.

$f_n(1) - f(1) = 0$ dosaď $x = \sqrt[n]{\frac{1}{2}}$

Máme $\sigma_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |f_n(x)| = \sup_{x\in[0,1)} |x^n| \geq \left(\frac{1}{\sqrt[n]{2}}\right)^n = 1/2$

tedy $\sigma_n \not\to 0$, tedy dle T 2.2: $f_n \not\rightrightarrows f$.

Theorem 2.3: Ať $\sum \sigma_n < \infty$, kde $\sigma_n = \sup_{x\in E} |f_n(x)|$.

Pak $\sum_{n=1}^{\infty} f_n \rightrightarrows$

Dk.:

FIX $\varepsilon > 0$.

[B-C podmínka: $x_n \to \iff \forall \varepsilon>0\ \exists n_0\ \forall m,n \geq n_0\ |x_n - x_m| < \varepsilon$]

Víme $\lim_{N\to\infty} \sum_{n=1}^{N} \sigma_n \in \mathbb{R}$

dle B-C podmínky $\exists N_0\ \forall M > N \geq N_0$: $\Big|\underbrace{\sum^N \sigma_n - \sum^M \sigma_n}_{= \sum_{n=N+1}^{M} \sigma_n}\Big| < \varepsilon$

$= \sum^N f_n(x) - \sum^M f_n(x)$

Tedy $\forall M > N \geq N_0$: $\left|\sum_{n=N+1}^{M} f_n(x)\right| \leq \sum_{n=N+1}^{M} |f_n(x)| \leq \sum_{n=N+1}^{M} \sigma_n < \varepsilon$

$\forall x \in E$

Tedy, dle B-C podmínky, $\lim_{N\to\infty} \sum^N f_n(x)$ existuje pro $x \in E$.

Označ $f(x) := \sum_{n=1}^{\infty} f_n(x)$.

Pak pro $N > N_0$ a $x \in E$ dokážeme

- NAJDU $M > N$ že

$$\left|\sum_{n=1}^{M} f_n(x) - f(x)\right| < \varepsilon$$

PAK

$$\left|\sum_{n=1}^{N} f_n(x) - f(x)\right| \leq \left|\sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{M} f_n(x)\right| + \left|\sum_{n=1}^{M} f_n(x) - f(x)\right| < 2\varepsilon.$$

PŘ

$$\sum_{n=1}^{\infty} \overbrace{\frac{n x}{1+n^5x^2}}^{=f_n(x)}, \quad x \in \mathbb{R}$$

BODOVÁ KCE: Pro $x \in \mathbb{R}$ řada absolutně konverguje.

Zkoumej

$$\lim_{n\to\infty} \frac{f_n(x)}{\frac{1}{n^4}} = \lim_{n\to\infty} n^4 \frac{n x}{n^5\left(x^2+\frac{1}{n^5}\right)} = \frac{1}{x} \in \mathbb{R}\setminus\{0\}$$

pro $x \neq 0$

$$\underset{\text{LSK + FAKT } \sum \frac{1}{n^4} < \infty}{\Longrightarrow} \quad \sum f_n(x) < \infty \text{ pro } x \neq 0$$

A pro $x=0$ máme $f_n(0) = 0$, tedy $\sum f_n(x) = 0$.

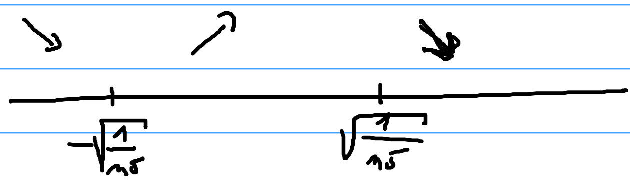
Závěr 1: Řada bodově konverguje.

STEJNOMĚRNÁ KCE:

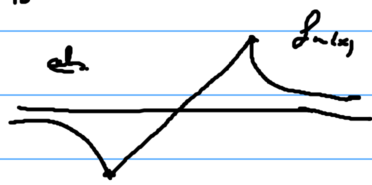
$$\sigma_n := \sup_{x\in\mathbb{R}} |f_n(x)|$$

$$f_n'(x) = \frac{n(1+n^5x^2) - nx(2xn^5)}{(\ldots)^2} = \frac{n(1+n^5x^2-2x^2n^5)}{(\ldots)^2}$$

$$= \frac{n(1-x^2n^5)}{(\ldots)^2}$$



$$\lim_{x\to\pm\infty} f_n(x) = \lim_{x\to\pm\infty} \frac{nx}{x^2\left(\frac{1}{x^2}+n^5\right)} = \underline{\underline{0}}.$$



$$\Rightarrow \sup_{x\in\mathbb{R}} |f_n(x)| = \max\left\{\left|f_n\left(\frac{1}{\sqrt{n^5}}\right)\right|, \left|f_n\left(-\frac{1}{\sqrt{n^5}}\right)\right|\right\}$$

$$= \frac{n\cdot n^{-5/2}}{1+n^5\cdot n^{-5}} = \frac{1}{2}\cdot n^{-3/2} = \frac{1}{2n^{3/2}}$$

Celkem: $\sigma_n = \sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{2n^{3/2}}$, tedy $\sum \sigma_n < \infty$

(PŘIPOMENUTÍ: $\sum \frac{1}{n^a} < \infty \iff a > 1$)

$$\underset{M\text{-test}}{\Longrightarrow} \quad \sum f_n \rightrightarrows$$

Tvrzení 2.4: (a) $f_n \rightrightarrows f$, f_n spojité $\Rightarrow$ f spojitá

(b) $\sum f_n \rightrightarrows f$, — " — $\Rightarrow$ — " —

DK NÍŽE

Př: DOKAŽTE, ŽE

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R} \qquad \text{JE SPOJ. FCE}$$

(POZN.: $f(x) = \exp(x)$)

⌈ Zvol $x_0 \in \mathbb{R}$. Pro $E := (x_0 - 1, x_0 + 1)$ máme

$$\sigma_n = \sup_{x \in E} \left| \frac{x^n}{n!} \right| = \frac{1}{n!} \max\{ |x_0+1|^n, |x_0-1|^n \} \le \frac{(|x_0|+1)^n}{n!}$$

a tedy $\sum \sigma_n < \infty$, protože víme: $\sum \frac{a^n}{n!} < \infty$ pro $a > 1$

⌈ PODÍLOVÉ KRIT.: $\frac{a_{n+1}}{a_n} \to\ < 1 \Rightarrow \sum a_n < \infty$ ⌋

A máme $\frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1} \to 0 < 1$

$$\overset{\text{M-test}}{\Longrightarrow} \sum \frac{x^n}{n!} \rightrightarrows \text{ na } E = (x_0 - 1, x_0 + 1)$$

$$\underset{\text{ZACHOVÁNÍ SPOJ.}}{\Longrightarrow} \sum \frac{x^n}{n!} \text{ je SPOJ na } E, \text{ speciálně v bodě } x_0$$

$x_0 \in \mathbb{R}$ byl libovolný $\Rightarrow$ $f(x)$ spojitá na $\mathbb{R}$ ⌋

Dk 2.4: (a) Fix $x \in E$, $\varepsilon > 0$. Zvol $n_0 \in \mathbb{N}$ tak že $\sup_{x \in E} |f_{n_0}(x) - f(x)| < \frac{\varepsilon}{3}$

Pro toto n_0 najdi $\delta > 0$: $|y - x| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(y)| < \varepsilon/3$

Pak $|y - x| < \delta \Rightarrow$

$$|f(y) - f(x)| \le |f_{n_0}(y) - f(y)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(x) - f(x)| < \varepsilon.$$

(b) plyne z (a) $\boxtimes$

Př: Ukažte, že pro $f(x)$ z minulého příkladu platí

$$f'(x) = f(x)$$

⌈ Máme pro $f_n(x) = \frac{x^n}{n!}$, že $f_n'(x) = \frac{x^{n-1}}{(n-1)!}$ pro $n \geq 1$

$$f_0(x) = 1 \qquad f_0'(x) = 0$$

Tedy
$$\sum_{n=0}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

$\underbrace{\text{VÍME (viz. minulý př.)}}$: $\sum \frac{x^n}{n!} \rightrightarrows$ na (x_0-1, x_0+1) pro $x_0 \in \mathbb{R}$

$$\overset{T\,2.6}{\Longrightarrow} \underbrace{f'(x) = \sum_{n=0}^{\infty} f_n'(x) = f(x)}_{\text{SPECIÁLNĚ } f'(x_0) = f(x_0)} \text{ na } (x_0-1, x_0+1) \text{ pro } x_0 \in \mathbb{R}$$

$\Rightarrow x_0 \in \mathbb{R}$ libovolné $\Rightarrow f'(x) = f(x),\ x \in \mathbb{R}$ ⌋

Př

$$f(x) := \sum_{n=1}^{\infty} (-1)^n \frac{\sin\left(1 + \frac{x}{n}\right)}{\sqrt{n}}, \quad x \in [-1,1]$$

- f je SPOJITÁ:

POLOŽ $f_n(x) = (-1)^n \frac{\sin\left(1 + \frac{x}{n}\right)}{\sqrt{n}}$

(Pak $\sigma_n := \sup_{x \in} |f_n(x)| \leq \frac{1}{\sqrt{n}}$, ale $\sum \frac{1}{\sqrt{n}} = +\infty$

... TAKTO M-test nelze použít)

(i) Pak $f_n'(x) = (-1)^n \frac{\cos\left(1 + \frac{x}{n}\right)}{n^{3/2}}$

odkud $\tilde{\sigma}_n := \sup_{x \in \mathbb{R}} |f_n'(x)| \leq \frac{1}{n^{3/2}}$,

tedy $\sum_{n=1}^{\infty} \tilde{\sigma}_n < \infty$ (S.K. + fakt, že $\sum \frac{1}{n^{3/2}} < \infty$)

$\overset{\text{M-test}}{\Longrightarrow} \sum_{n=1}^{\infty} f_n'$ je $\rightrightarrows$-ální na $\mathbb{R}$

(ii) Pro $x=0$ máme

$$\sum_{n=1} f_n(0) = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(1)}{\sqrt{n}} \left(= \lim_{N \to \infty} \sum_{n=1}^{N} \dots \right)$$

$$\overset{\text{VAL}}{=} \sin(1) \cdot \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}}_{\text{KJE DLE LEIBNITZ. KRIT.}} < \infty$$

$$\overset{T\,2.6}{\Longrightarrow} \quad f(x) \text{ je enakomerna limita, tj.}$$

$$\sum f_n(x) \rightrightarrows f \quad \text{na } \mathbb{R}$$

$$\overset{T\,2.4}{\Longrightarrow} \quad f \text{ je zvezna na } \mathbb{R}.$$

<u>Izračun $f'(0)$</u>: po T 2.6 imamo

$$f'(0) = \sum_{n=1}^{\infty} f_n'(0) = \underline{\sum_{n=1}^{\infty} (-1)^n \frac{\cos(1)}{n^{3/2}}}.$$

<u>PR</u>

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x \in (1,\infty).$$

Dokažite, da $f \in \mathcal{C}^1((1,\infty))$

Rešitev: $f_n(x) = \frac{1}{n^x} \left(= \exp(-x \log n) \right)$

Naj bo $x_0 > 1$:

(iii) $\cdot\ f_n'(x) = \frac{1}{n^x}(-\log n), \quad x > 1$ $\qquad$ a izberemo tako, da $x_0 - a > 1$

$$\sigma_n = \sup_{x > x_0} |f_n'(x)| = |f_n'(x_0)| = \frac{\log n}{n^{x_0}} \leq \frac{n^a}{n^{x_0}} = \frac{1}{n^{x_0 - a}}$$

$$\left(\lim_{n\to\infty} \frac{\log n}{n^a} = 0 \text{ za } a > 0, \text{ tj. } \forall a > 0\ \exists n_0\ \forall n \geq n_0 \quad \log n \leq n^a \right)$$

$$\Longrightarrow \sum \sigma_n < \infty \quad \left(\text{S.K.} + \text{FAKT, da } \sum \frac{1}{n^{x_0 - a}} < \infty \text{ ker } x_0 - a > 1 \right)$$

$$\overset{\text{W-test}}{\Longrightarrow} \sum f_n'(x) \text{ je } \rightrightarrows \text{ enakomerna na } (x_0, \infty).$$

(ii) Máme $\sum f_n(x) < \infty$ pro každé $x > 1$.

$\overset{T2.6}{\Longrightarrow}$ $\sum f_n(x) \rightrightarrows$ na (x_0, ∞) pro každé $x_0 > 1$

a na (x_0, ∞) máme $\forall x \in (x_0, \infty)$!

$$f'(x) = \sum f_n'(x) = \sum_{n=1}^{\infty} \frac{-\log n}{n^x}$$

x_0 bylo libovolné, tedy

$$f'(x) = \sum f_n'(x), \quad x \in (1, \infty).$$

Zároveň, pro každé $x_0 > 1$ je $\sum f_n' \rightrightarrows$ na (x_0, ∞)

$\overset{T2.4}{\Longrightarrow}$ f' je spojitá na (x_0, ∞) pro každé $x_0 > 1$

$\Longrightarrow$ f' je spojitá na $(1, \infty)$.

PŘ: $f(x) := \sum_{n=1}^{\infty} \frac{|x|}{n^2+x^2}, \quad x \in [-1,1]$.

• $f \in \mathcal{C}([-1,1])$: Položme $f_n(x) := \frac{|x|}{n^2+x^2} \left(= \frac{\operatorname{sgn}(x)\cdot x}{n^2+x^2}\right)$

$$\sigma_n = \sup_{x\in(-2,2)} |f_n(x)| \le \frac{2}{n^2}, \quad \text{tedy } \sum \sigma_n < \infty$$

(s.k. + fakt, že $\sum \frac{1}{n^2} < \infty$)

$\overset{W\text{-test}}{\Longrightarrow}$ $\sum f_n \rightrightarrows$ na $(-2,2) \supseteq [-1,1]$

$\overset{T2.4}{\Longrightarrow}$ f je spojitá na $[-1,1]$ (dokonce i na $(-2,2)$)

• f_n' (1/2): $f_n'(x) = \operatorname{sgn}(x)\cdot \frac{(n^2+x^2) - x(2x)}{(n^2+x^2)^2} = \operatorname{sgn}(x)\frac{n^2-x^2}{(n^2+x^2)^2}, \; x \neq 0$

(iii) [$f_n'(0)$ neex.]

$$\sigma_n = \sup_{x\in(0,1]} |f_n'(x)| \le \frac{n^2}{n^4} = \frac{1}{n^2}, \quad \sum \sigma_n < \infty$$

(s.k. + fakt, že $\sum \frac{1}{n^2} < \infty$)

$\overset{W\text{-test}}{\Longrightarrow}$ $\sum f_n' \rightrightarrows$ na $(0,1]$

$$\sum f_n(x) < \infty \quad \text{na} \quad [-1,1]$$

$$\overset{T2.6}{\Longrightarrow} \quad f'(x) = \sum f_n'(x) \quad , \quad x \in (0,1]$$

Speciálně, $$f'(1/2) = \sum f_n'(1/2) = \sum_{n=1}^{\infty} \frac{n^2 - 1/4}{(n^2+\frac{1}{4})^2}$$

• $f'(0)$:

Máme:

$$f_+'(0) = \lim_{x\to 0+} \frac{f(x) - \overbrace{f(0)}^{=0}}{x-0} = \lim_{x\to 0+} \frac{1}{x} \sum_{n=1}^{\infty} f_n(x)$$

$$= \lim_{x\to 0+} \sum_{n=1}^{\infty} \frac{\operatorname{sgn}(x)}{n^2+x^2} = \sum_{n=1}^{\infty} \lim_{x\to 0+} \frac{\operatorname{sgn}(x)}{n^2+x^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

POUŽIJI T2.5

Víš: $\sum \frac{\operatorname{sgn}(x)}{n^2+x^2} \rightrightarrows$ na $[-1,1]$

$$\sigma_n = \sup_{x\in[-1,1]} \left|\frac{\operatorname{sgn}(x)}{n^2+x^2}\right| \le \frac{1}{n^2} \quad , \quad \sum \sigma_n < \infty \left(\text{s. k.} + \sum \frac{1}{n^2} < \infty \right)$$

$$\overset{W-test}{\Longrightarrow} \quad \sum \frac{\operatorname{sgn}(x)}{n^2+x^2} \rightrightarrows \text{na} \ [-1,1]$$

Analogicky: $$f_-'(0) = -\sum \frac{1}{n^2}$$

$\Longrightarrow$ $f'(0)$ neex.

Př.

$$f(x) := \sum_{n=1}^{\infty} (-x^2+6x-8)^n$$

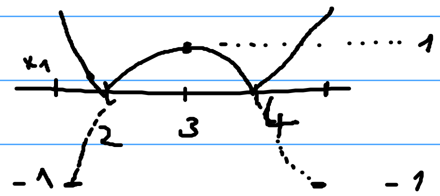
• D_f (tj. x že $f(x)\in\mathbb{R}$):

Pozn!: $\sum q^n < \infty \iff q\in(-1,1)$

Tedy $f(x)\in\mathbb{R} \iff |-x^2+6x-8| < 1$

$$|-(x-2)(x-4)| < 1$$

obr.:



$\iff x\in(x_1,3)\cup(3,x_2)$

kde $x_1\in(-\infty,2)$,

$x_2\in(4,\infty)$

a $-x_i^2+6x_i-8=-1$

$i\in\{1,2\}$

Řešme: $-x^2+6x-8=-1$

$-x^2+6x-7=0$ $\qquad x_{1,2} = \frac{-6\pm\sqrt{36-28}}{-2}$

$$= 3\pm\frac{\sqrt{8}}{2} = 3\pm\sqrt{2}$$

$\Rightarrow$ tj. $\boxed{f(x)\in\mathbb{R} \iff x\in(3-\sqrt{2},3)\cup(3,3+\sqrt{2})}$

• f je spoj. v $7/2$: $\quad 7/2\in[3+\varepsilon,4]$ pro $\varepsilon=1/4$ (třeba)

Položme $E:=[3+\varepsilon,4]$. Pak

$$\sigma_n = \sup_{x\in E} |(-x^2+6x-8)^n| = |\underbrace{-(3+\varepsilon)^2 - 6\cdot(3+\varepsilon)-8}_{=:q<1}|^n, \quad \sum\sigma_n<\infty$$

$\xRightarrow{W\text{-test}}$ $\sum(-x^2+6x-8)^n \rightrightarrows$ na $E=[3+\varepsilon,4]$

$\xRightarrow{T\,2.4}$ $f(x)$ je spojitá na $(3+\varepsilon,4)$, speciálně v bodě $7/2$.

• $f'(7/2)$: $\quad f_n(x) = (-x^2+6x-8)^n$

$$f_n'(x) = n(-x^2+6x-8)^{n-1}\cdot(-2x+6)$$

iii)

$$\sigma_n = \sup_{x\in E} |f_n'(x)| \underset{\text{viz. výše}}{\leq} n\cdot q^{n-1}\cdot \max_{x\in E} |-2x+6|$$

$$= n\cdot q^{n-1}\cdot \max\{\overbrace{|-2(3+\varepsilon)+6|}^{-2\varepsilon^n}, \overbrace{|-8+6|}^{{}_{\shortparallel}2}\}$$

$$\underset{\varepsilon = 1/4}{=} 2n\, q^{n-1}, \quad \text{tedy } \sum \sigma_n < \infty$$

S. k. + Fakt, že $\sum n q^n < \infty$ neboť

$$\lim_{n\to\infty} \sqrt[n]{nq^n} = q \lim_{n\to\infty} \sqrt[n]{n} = q < 1$$

$$\overset{\text{ODMOCNIN. KRIT.}}{\Longrightarrow} \quad \sum n q^n < \infty$$

$$\Longrightarrow \frac{2}{q}\sum n q^n < \infty$$

$$\shortparallel$$

$$\sum 2n\, q^{n-1}$$

$$\overset{\text{W-test}}{\Longrightarrow} \sum f_n'(x) \rightrightarrows \text{ na } E = [3+\varepsilon, 4]$$

ii) $\sum f_n(x)$ je konv. na E (viz. výše)

$$\overset{T2.6}{\Longrightarrow} f'(x) = \sum f_n'(x) \text{ na } (3+\varepsilon, 4)$$

Speciálně, $\quad f'(7/2) = \sum_{n=1}^{\infty} f_n'(7/2) = \text{DOSADÍ}\ldots$

MOCNINNÉ ŘADY

Př

$$\sum_{n=0}^{\infty} \frac{n^7}{n+20}\, x^n \qquad \left[\, a = 0\ ;\ a_n = \frac{n^7}{n+20} \,\right]$$

Výpočet poloměru konv.:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^7}{\sqrt[n]{n\left(1+\frac{20}{n}\right)}} \overset{AL+Z.L.\ (\sqrt[n]{n}\to 1)}{=}$$

$$= \frac{1^7}{1 \cdot \lim\limits_{n\to\infty} \sqrt[n]{1+\frac{20}{n}}} \overset{(*)}{=} 1$$

$(*)$

$$1 \leftarrow \sqrt[n]{1} \le \sqrt[n]{1+\frac{20}{n}} \le \sqrt[n]{2} \longrightarrow 1 \qquad \left(\sqrt[n]{a} \longrightarrow 1 \ \text{ pro } a>0\right)$$

$$\overset{\text{POLICIE}}{\Longrightarrow} \quad \lim_{n\to\infty} \sqrt[n]{1+\frac{20}{n}} = 1$$

$\Rightarrow R = 1$ je poloměr konvergence řady.

Tedy: pro $|x| < 1$ ... řada AK

$|x| > 1$ ... řada D

Pro $x = 1$: $\sum \frac{n^7}{n+20}$ ... řada D., neboť $\frac{n^7}{n+20} \to \infty \neq 0$

(není splněna nutná podmínka)

Pro $x = -1$: $\sum \frac{n^7}{n+20}(-1)^n$ ... řada D., neboť její členy nekonvergují k nule

(pozn.: $a_n \to 0 \iff |a_n| \to 0$)

Př

$$\sum \frac{x^n}{n^p} \qquad (p \in \mathbb{R})$$

Výpočet poloměru konv.:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^p} \overset{Z.L.+\text{SPOJITOST } (\cdot)^p}{=} 1$$

$\Rightarrow R = 1$ je poloměr kce

Tedy: pro $|x| < 1$ ... řada AK

$|x| > 1$ ... řada D

Pro $x=1$: $\sum \frac{1}{n^p}$ • AK $\Leftrightarrow$ $p > 1$ $\Leftrightarrow$ K

• D $\Leftrightarrow$ $p \leq 1$

Pro $x = -1$: $\sum \frac{(-1)^n}{n^p}$ • AK $\Leftrightarrow$ $p>1$ (viz. $\uparrow$)

• LEIBNITZ:

K $\Leftarrow$ $p \in (0,1]$

(ale ne AK)

• Pro $p \leq 0$ ... řada D. podle nutné podmínky

(členy „nejdou k nule")

PŘ

$$\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n} (x+1)^n \qquad \left[a = -1;\ a_n = \frac{3^n + (-2)^n}{n}\right]$$

Výpočet poloměru kce:

$$\lim_{n\to\infty} \sqrt[n]{\frac{3^n + (-2)^n}{n}} \overset{AL + 2.L.\ (\sqrt[n]{n} \to 1)}{=} \lim_{n\to\infty} \sqrt[n]{3^n\left(1 + \left(-\tfrac{2}{3}\right)^n\right)}$$

$$\overset{AL}{=} 3 \lim_{n\to\infty} \sqrt[n]{1 + \left(-\tfrac{2}{3}\right)^n} \overset{(*)}{=} 3$$

$$\left[(*)\quad 1 \leftarrow \sqrt[n]{1} \leq \sqrt[n]{1 + \left(-\tfrac{2}{3}\right)^n} \leq \sqrt[n]{2} \to 1 \qquad \overset{POLICE}{\Rightarrow} \sqrt[n]{1+\left(-\tfrac{2}{3}\right)^n} \to 1 \right]$$

$\Rightarrow R = \frac{1}{3}$ je poloměr kce

Tedy: pro $x \in \left(-\frac{4}{3}, -\frac{2}{3}\right)$ ... řada AK

$x < -4/3$ nebo $x > -2/3$ ... řada D

Pro $x = -\frac{2}{3}$:

$$\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n} \left(\frac{1}{3}\right)^n$$

$$\overset{AL}{=} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{D} + \underbrace{\sum_{n=1}^{\infty} \frac{(-2/3)^n}{n}}_{AK} \quad \Delta$$

$\left(\circledast \text{ Máme } \lim_{n\to\infty} \sqrt[n]{\frac{(2/3)^n}{n}} = 2/3 < 1 \overset{\text{odmocninové krit.}}{\Longrightarrow} \sum \frac{(2/3)^n}{n} < \infty \right)$

$\Rightarrow$ Pro $x = -\frac{2}{3}$ je řada divergentní

Pro $x = -4/3$:

$$\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n} \left(-\frac{1}{3}\right)^n \overset{AL}{=} \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n}}_{\text{k. dle Leibnize}} + \underbrace{\sum_{n=1}^{\infty} \frac{(2/3)^n}{n}}_{\text{k (viz. } \circledast)}$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \cdots + \cdots = \sum^{N} \cdots + \sum^{N} \cdots$$

$\Rightarrow$ Pro $x = -4/3$ je řada konv. (ale ne AK – viz.)

PŘ

$$\sum_{n=1}^{\infty} \frac{(3 + (-1)^n)^n}{n} x^n$$

⌈ Výpočet poloměru konv.:

$$\limsup_{n\to\infty} \sqrt[n]{\frac{(3+(-1)^n)^n}{n}} \overset{AL + Z.L.}{=} \limsup_{n\to\infty} 3 + (-1)^n = 4$$

$\Rightarrow R = \frac{1}{4}$ je poloměr konv. této řady

Tedy:

pro $|x| < 1/4$ řada AK

$|x| > 1/4$ řada D

Pro $x = 1/4$:

$$\sum \frac{(3 + (-1)^n)^n}{n} \left(\frac{1}{4}\right)^n \overset{AL}{=} \underbrace{\sum_{n=1}^{\infty} \frac{4^{2n}}{2n} \frac{1}{4^{2n}}}_{D.} +$$

$$+ \sum_{n=0}^{\infty} \frac{2^{2n+1}}{2n+1} \frac{1}{4^{2n+1}} = \underbrace{\cancel{\cdots}}_{D.}$$

$$= \infty + \underbrace{\sum \frac{2^{2n+1}}{2n+1} \frac{1}{4^{2n+1}}}_{\geq 0} = +\infty$$

Pro $x = -1/4$ .... podobně se ukáže, že řada D ... ⌋

APLIKACE věty 2.9

Př.: $\sum \frac{n^n}{n!} x^n$ ... $R = \dfrac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}}} \overset{AL}{=} \lim\limits_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = ?$

Jinak: $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^n\,(n+1)}{(n+1)!} \cdot \dfrac{n!}{n^n}$

$= \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$

Věta 2.9 $\Rightarrow$ $R = \dfrac{1}{e}$

$\Rightarrow$ řada AK pro $|x| < 1/e$, D pro $|x| > 1/e$

(POZN.: pro $|x| = 1/e$ to je náročnější...)

(Věta 2.10 – DŮKAZ)

- VYNECHÁME VÝPOČET POLOMĚRU KCE

„IDEA": • $\lim\limits_{n\to\infty} \sqrt[n]{n\, a_n} \overset{AL}{=} \lim\limits_{n\to\infty} \sqrt[n]{a_n} = R$

• $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{a_n}{n+1}} = \dots R$

PRO LIB. $\sum a_n (x-a)^n$ kde R je poloměr kce

- Pro $x_0 \in (a-R, a+R)$ ... $\exists \varepsilon > 0$: $[x_0-\varepsilon, x_0+\varepsilon] \subseteq (a-R, a+R)$

a řada $\sum\limits_{n=0}^{\infty} a_n (x-a)^n$ je $\rightrightarrows$ na $[x_0-\varepsilon, x_0+\varepsilon]$

Dk.: $\sigma_n := \sup\limits_{x\in[x_0-\varepsilon, x_0+\varepsilon]} |a_n (x-a)^n| = |a_n| \underbrace{\max\{|x_0-\varepsilon-a|, |x_0+\varepsilon-a|\}^n}_{=:}$

$= |a_n|\, |a-q|^n$

$q \in \{x_0-\varepsilon, x_0+\varepsilon\}$, tj. $|a-q| < R$

tedy $\sum \sigma_n = \sum |a_n|\, |a-q|^n < \infty$

$q \in (a-R, a+R)$, tj.

kce z definice R

$\overset{\text{m-teh}}{\Rightarrow} \quad \sum a_n (x-a)^n \rightrightarrows$ на $[x_0-\varepsilon, x_0+\varepsilon]$

Специални:

- $\sum n\, a_n (x-a)^{n-1}$ jo $\rightrightarrows$ на $[x_0-\varepsilon, x_0+\varepsilon]$

$$\overset{2.6}{\Rightarrow} \left(\sum_{n=0}^{\infty} a_n (x-a)^n \right)' = \sum_{n=1}^{\infty} n\, a_n (x-a)^{n-1}$$

- A също

$$\left(\sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1} \right)' = \sum_{n=0}^{\infty} a_n (x-a)^n$$

for $x \in [x_0-\varepsilon, x_0+\varepsilon]$

Специални $\quad f'(x_0) = \sum_{n=1}^{\infty} a_n \cdot n \cdot (x_0-a)^{n-1}$

a $\quad \left(\sum \frac{a_n}{n+1} (x_0-a)^{n+1} \right)' = f(x_0)$ $\square$ $\quad x_0 \in (a-R, a+R)$ libovolno

Pr

$$e^y = \sum \frac{y^n}{n!}, \quad y \in \mathbb{R}$$

a)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}}}$$

b)

$(1+x) \log(1+x)$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1 \ ; \quad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \left(= \frac{1}{1-x} - 1 \right)}$$

$$(\log(1+x))' = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n, \quad |x|<1$$

$$\overset{2.10}{\Rightarrow} \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \text{const}$$

Dosadō $x = 0$ ... $0 = \log(1+0) = 0 + \text{const.} \Rightarrow \text{const.} = 0$

$$\Rightarrow \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad |x| < 1$$

$$\Rightarrow (1+x)\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+2}$$

= .... posčítáme mocniny u stejných mocnin „x"

$$= x + \sum_{n=2}^{\infty} \left(\frac{(-1)^{n-1}}{n} + \frac{(-1)^{n-2}}{n-1} \right) x^n, \quad |x| < 1$$

SČÍTÁNÍ MOCNINNÝCH ŘAD

PŘ $\displaystyle\sum_{n=0}^{\infty} \frac{x^{4n+5}}{n!}$ [PŘIPOMENÍ: $\exp(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ pro $x \in \mathbb{R}$]

$$= x^5 \sum_{n=0}^{\infty} \frac{(x^4)^n}{n!} = \underline{x^5 e^{x^4}}, \quad x \in \mathbb{R}$$

PŘ $\displaystyle\sum_{n=1}^{\infty} n \cdot x^n$

$$R = \frac{1}{\lim \sqrt[n]{n}} \overset{z.l.}{=} 1 \Rightarrow \text{ŘADA má poloměr konv. } R = 1,$$

AK pro $|x| < 1$, D jinak

(protože pro $x = 1$ nebo $x = -1$ jejich členy nekonvergují k nule)

$\Rightarrow$ Chceme najít součet pro $|x| < 1$.

[PŘIPOMENÍ: $\displaystyle\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, $\displaystyle\sum_{n=1}^{\infty} q^n = \frac{q}{1-q}$ pro $|q| < 1$]

Máme
$$\sum_{n=1}^{\infty} n\, x^n = x \left(\sum_{n=1}^{\infty} n \cdot x^{n-1} \right)$$
$$= x \cdot \left(\sum_{n=0}^{\infty} x^n \right)' = x \cdot \left(\frac{1}{1-x} \right)'$$
$$= \underline{x \cdot \frac{1}{(1-x)^2}}, \quad |x| < 1$$

PŘ: $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n(n+1)}}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n^2}\,\sqrt[n]{1+\frac{1}{n}}}} \overset{\circledast}{\underset{AL + z.l. + \circledast}{=}} 1$$

[$\circledast$ $\underset{\downarrow\, 1}{\sqrt[n]{1}} \leq \sqrt[n]{1+\frac{1}{n}} \leq \underset{\downarrow\, 1}{\sqrt[n]{2}}$ $\overset{\text{POLICIE}}{\Longrightarrow}$ $\sqrt[n]{1+\frac{1}{n}} \to 1$]

$\Rightarrow$ Řada [illegible] pro $|x| < 1$.

Máme

$$\bullet \quad \left(\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} \right)' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n+1} \overset{x \neq 0}{=} \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = \triangle$$

$$\bullet \quad \left(\sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} \right)' = \sum_{n=1}^{\infty} x^n \overset{|x|<1}{=} \frac{x}{1-x}$$

$$\int \frac{x}{1-x}\, dx = \int \frac{-(1-x)}{1-x} + \frac{1}{1-x}\, dx \overset{C}{=} -x - \log(1-x)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = -x - \log(1-x) + C, \quad |x| < 1$$

Dosaď $x = 0$ ... $0 = -0 - 0 + C \Rightarrow C = 0$

$$\Rightarrow \underline{\underline{\sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = -x - \log(1-x), \quad |x| < 1}}$$

$$\triangle = \frac{1}{x^2}\left(-x - \log(1-x)\right) = -\frac{1}{x} - \frac{\log(1-x)}{x^2}$$

$$\int \frac{1}{x} + \frac{\log(1-x)}{x^2}\, dx = \log|x| + \int \frac{\log(1-x)}{x^2}\, dx$$

$$= \left| \begin{array}{lll} \underline{\text{PER PARTES}}: & u' = \frac{1}{x^2} & v = \log(1-x) \\ & u = -\frac{1}{x} & v' = -\frac{1}{1-x} \end{array} \right|$$

$$= \log|x| - \frac{\log(1-x)}{x} - \int \underbrace{\frac{1}{x(1-x)}}_{= \frac{1}{x} + \frac{1}{1-x}} dx$$

$$\overset{C}{=} \cancel{\log|x|} - \frac{\log(1-x)}{x} \cancel{- \log|x|} + \log(1-x)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \begin{cases} -\log(1-x) + \frac{\log(1-x)}{x} + C, & |x| < 1,\ x \neq 0 \\ 0, & x = 0 \end{cases}$$

Navíc, $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ je spojitá na $(-1, 1)$.

Tedy:
$$0 = \lim_{x\to 0} -\lg|1-x| + \frac{\lg|1-x|}{x} + c \overset{AL + Z.L}{=} -0 - 1 + c$$

$$\left(\lim_{x\to 0} \frac{\lg|1-x|}{x} \overset{L'H\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{-1}{1-x} = -1 \right)$$

$$\Rightarrow \underline{\underline{c = 1}}$$

Závěr:
$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \begin{cases} 0 & \text{--- } x = 0 \\ 1 - \lg|1-x| + \frac{\lg|1-x|}{x}, & |x|<1,\ x\neq 0 \end{cases}$$

Pro $x = 1$ a pro $x = -1$:

Řada je AK

$$\sum \left| \frac{x^n}{n(n+1)} \right| \overset{\text{pro } |x|=1}{=} \sum \frac{1}{n(n+1)} < \infty$$

$$\lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = 1 \Rightarrow \text{LSK + FAKT, že } \sum \frac{1}{n^2} < \infty$$

$\Rightarrow$ dle Abelovy věty:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{x\to 1^-} \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \lim_{x\to 1^-} 1 - \lg|1-x| + \frac{\lg|1-x|}{x}$$

$$\overset{AL}{=} 1 + \lim_{x\to 1^-} \frac{1}{x}\left(\lg(1-x)(1-x)\right) \overset{AL + VOLSF}{=} 1 + \lim_{y\to 0^+} \lg y \cdot y = \underline{\underline{1}}$$

Z.L.: $\lim_{y\to 0^+} y^\alpha \lg y = 0, \quad \alpha > 0$

Analogicky:
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = \lim_{x\to (-1)^+} 1 - \lg|1-x| + \frac{\lg|1-x|}{x} = 1 - \lg(2) - \lg(2)$$
$$= \underline{1 - 2\lg 2}.$$

PŘ
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right)' = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n \overset{|x|<1}{=} \frac{1}{1-x}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n} = -\lg|1-x| + c \quad , |x| < 1$$

Dosaď $x=0$: $\quad 0 = -0 + c \implies c=0$

$$\implies \sum_{n=1}^{\infty} \frac{x^n}{n} = -\lg(1-x), \quad |x|<1$$

Řada $\sum \frac{(-1)^n}{n}$ je konv. dle Leibnizova krit.

$$\overset{\text{Abel}}{\implies} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \lim_{x\to(-1)^+} \sum_{n=1}^{\infty} \frac{x^n}{n} = \lim_{x\to(-1)^+} -\lg(1-x)$$

$$= \underline{\underline{-\lg 2}}$$

Př: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\sum_{n=1}^{\infty} n x^n \overset{\text{viz. Př V.3.6}}{=} \frac{x}{(1-x)^2}, \quad |x|<1$$

$$\underset{\text{dosaď } x=\frac{1}{2}}{\implies} \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2}\cdot 4 = \underline{\underline{2}}.$$

Př: $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2n+3}{(n+1)2^n}$

$f(x) := \displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{n+1} x^n$ .... má poloměr konv. $R=1$

(neboť $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{2n+3}{n+1}} \overset{(*)}{=} 1$)

$\Big[$ (*) $\displaystyle\lim_{n\to\infty} \frac{2n+3}{n+1} \overset{\text{AL}}{=} 2$, tedy $\exists n_0\ \forall n \geq n_0: \underset{1}{\sqrt[n]{1}} \leq \sqrt[n]{\frac{2n+3}{n+1}} \leq \sqrt[n]{3}$

$\overset{\text{POLICIE}}{\implies} \displaystyle\lim \sqrt[n]{\frac{2n+3}{n+1}} = 1 \Big]$

$\implies f(x)$ je dobře def. pro $|x|<1$

$$f(x) \overset{x\neq 0}{=} \frac{1}{x} \underbrace{\sum_{n=1}^{\infty} \frac{2n+3}{n+1} x^{n+1}}_{=:g(x)}$$

- $\displaystyle g'(x) = \sum_{n=1}^{\infty} (2n+3) x^n = 2\sum_{n=1}^{\infty} n x^n + 3\sum_{n=1}^{\infty} x^n$

$$\underset{\text{viz. V.3.6}}{=} \frac{2x}{(1-x)^2} + \frac{3x}{1-x}, \quad |x|<1$$

$$= \frac{2(x-1)}{(1-x)^2} + \frac{3(x-1)}{1-x} + \frac{2}{(1-x)^2} + \frac{3}{1-x}$$

$$= -\frac{2}{1-x} - 3 + \frac{2}{(1-x)^2} + \frac{3}{1-x}$$

$$= -3 + \frac{2}{(1-x)^2} + \frac{1}{1-x}$$

$$\Rightarrow g(x) = \int -3 + \frac{2}{(1-x)^2} + \frac{1}{1-x}\, dx =$$

$$= -3x + 2\,\frac{1}{1-x} - \log(1-x) + c$$

DOSAĎ $x=0$:

$$0 = g(0) = -0 + 2 - 0 + c \Rightarrow \underline{\underline{c=-2}}$$

$$\Rightarrow g(x) = -3x + \frac{2}{1-x} - \log(1-x) - 2, \quad |x|<1$$

$$\Rightarrow f(x) = \frac{1}{x} g(x), \quad |x|<1, \quad x \neq 0$$

Závěr:

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n+3}{(n+1)2^n} = f(-\tfrac{1}{2}) = -2\, g(-\tfrac{1}{2})$$

$$= -2\left(\frac{3}{2} + \frac{4}{3} - \log\left(\frac{3}{2}\right) - 2\right)$$

$$\frac{3}{2} + \frac{4}{3} - 2 = \frac{9+8-12}{6}$$

$$= \underline{\underline{2\log(3/2) - \frac{5}{3}}}.$$

APLIKACE MOCNINNÝCH ŘAD NA ŘEŠENÍ ODR

Př.:

$$y'(x) = 3y(x/2), \quad y(0) = 1$$

Ř.: $y(x) = \sum_{n=0}^{\infty} a_n x^n$, pak $y(0) = a_0 = 1$

dále PRO $|x|<R$, R je poloměr konv. mocninné řady

$$y'(x) = 3y(x/2) \quad \& \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \text{ tj. } y'(0) = a_1$$

$$\Rightarrow \text{tj. } \underset{\substack{\| \\ a_1}}{y'(0)} = 3y(0) = 3 \quad \Rightarrow \underline{\underline{a_1 = 3}}$$

DERIVUJ

$$y''(x) = 3y'(x/2)\cdot\frac{1}{2} \quad \& \quad y''(0) = 2\cdot a_2 \quad \Rightarrow \underline{\underline{a_2 = \tfrac{1}{4}\,3\,y'(0) = \tfrac{9}{4}}}$$

$$\left(\Rightarrow y''(0) = \frac{3y'(0)}{2}\right)$$

$\vdots$

$$y^{(n+1)}(x) = 3\, y^{(n)}\left(\tfrac{x}{2}\right) \cdot \left(\tfrac{1}{2}\right)^n \quad \& \quad y^{(n+1)}(0) = (n+1)!\, a_{n+1}$$

$$\hookrightarrow \text{tj. } y^{(n+1)}(0) = 3\, y^{(n)}(0)\left(\tfrac{1}{2}\right)^n = 3\, n!\, a_n \left(\tfrac{1}{2}\right)^n$$

$$\Longrightarrow \underline{\underline{a_{n+1}}} = \frac{1}{(n+1)!}\, 3 \cdot n! \cdot \left(\tfrac{1}{2}\right)^n \cdot a_n = \underline{\underline{\frac{3}{n+1} \cdot \left(\tfrac{1}{2}\right)^n \cdot a_n}}$$

<u>Celkem</u>: Pokud $y(x) = \sum_{n=0}^{\infty} a_n x^n$ a platí:

- $R > 0$ (nulová lze [illegible])
- $$\begin{cases} a_0 = 1 \\ a_{n+1} = \dfrac{3}{n+1} \cdot \left(\tfrac{1}{2}\right)^n \cdot a_n, & n \geq 0 \end{cases}$$

Pak $y(x)$ řeší zadanou ODR.

~~<u>Zobecnění</u>: POKUD (ax) pro 2x~~

DOKONČENÍ PŘÍKLADU:

$$y'(x) = 3y\left(\tfrac{x}{2}\right), \quad y(0) = 1$$

MINULE: POKUD $y(x) = \sum a_n x^n$ na okolí 0:

$$(n+1)!\, a_{n+1} = y^{(n+1)}(0) = \left(3y\left(\tfrac{x}{2}\right)\right)^{(n)}(0) = 3\cdot\left(\tfrac{1}{2}\right)^{n} y^{(n)}(0)$$

$\rightsquigarrow$ VÍME:

$$\begin{cases} a_0 = 1 \\ a_{n+1} = \dfrac{3}{n+1}\left(\dfrac{1}{2}\right)^{n+1} a_n, & n \geq 1 \end{cases}$$

PAK $\sum a_n x^n$ ~~ŘEŠÍ~~ ZADANOU RCI, POKUD $R > 0$

DNES:

POLOŽME $a_n = \dfrac{3^n}{n!\; 2^{1+\dots+n}} = \dfrac{3^n}{n!\; 2^{n(n-1)/2}}$

PAK

- $a_0 = 1$ (DOSADÍME)
- $a_{n+1} = \dfrac{3^n\, 3}{(n+1)\underbrace{n!\; 2^{1+\dots+n}}_{a_n}\, 2^{n+1}} = \dfrac{3}{n+1}\left(\dfrac{1}{2}\right)^{n+1} a_n$

ZÁROVEŇ

$$\lim_{n\to\infty} \sqrt[n]{a_n} \overset{AL}{=} \frac{3}{2}\lim_{n\to\infty} \sqrt[n]{\frac{1}{n!\; 2^{1+\dots+n-1}}} \overset{(*)}{=} 0$$

Máme

$$(*) \quad 0 \leq \sqrt[n]{\frac{1}{n!\;\underbrace{2^{1+\dots+n-1}}_{\geq 1}}} \leq \sqrt[n]{\frac{1}{n!}} \longrightarrow 0$$

$\lim \sqrt[n]{n!} = +\infty$ neboť

$$n! = n(n-1)(n-2)\dots\left\lfloor\tfrac{n}{2}\right\rfloor \geq \left(\left\lfloor\tfrac{n}{2}\right\rfloor\right)^{n/2}$$

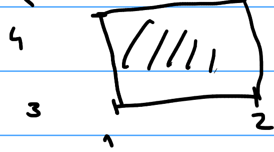
$$\Rightarrow \sqrt[n]{n!} \geq \sqrt[n]{\left(\left\lfloor\tfrac{n}{2}\right\rfloor\right)^{n/2}} = \sqrt{\left\lfloor\tfrac{n}{2}\right\rfloor} \longrightarrow \infty \quad \Rightarrow \text{POLICIE} \quad \lrcorner$$

(*) $= 0$ z věty o policii ∧ z (Δ)

$\Rightarrow$ poloměr konv. řady $\sum a_n x^n$ je $R = +\infty$.

TEORIE MÍRY

Motivace: • $\lambda^2\left([1,2]\times[3,4]\right) = 1$.



• Přirozená definice obsahu pro „obdélníky"

• Pro jiné objekty

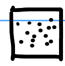
Pokud $\lim_{\varepsilon\to 0}$ obsahy vnějších obdélníků

$= \lim_{\varepsilon\to 0}$ obsahy „vnitřních" obdélníků

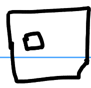
Pak to je obsah (tzv. Jordan–Peanův obsah)

Takováto def. ale dobře nefunguje:

Např.:



$$(J\text{-}P)^*\left([0,1]^2\setminus Q^2\right) = 1 \neq (J\text{-}P)_*\left([0,1]^2\setminus Q^2\right) = 0$$



$\Rightarrow$ definice obsahu která je rozumná

Důkaz faktu 3.1: dokážeme pro algebru (pro σ-algebru analogické)

Ať $A_1,\dots,A_m \in \mathcal{A}$, pak

$$\bigcap_{i=1}^{n} A_i \underset{\text{de Morgan}}{=} X\setminus \bigcup_{i=1}^{n} \underbrace{(X\setminus A_i)}_{\in \mathcal{A}} \in \mathcal{A}$$

⊠

(PŘ) X je množina, pak a) $\mathcal{A} = \{\emptyset, X\}$ ... pak $(X, \mathcal{A})$ je měř. prostor

b) $\mathcal{A} = \mathcal{P}(X) := \{ Y ; Y \subset X \}$

... pak $(X, \mathcal{A})$ je měř. prostor

c) $X = \mathbb{R}$, $\mathcal{A} = \{$OT. INTERVALY$\}$

.... pak $(X, \mathcal{A})$ není měř. prostor

totiž $\mathbb{R} \setminus (0,1)$ není ot. interval

PŘ: • $\{(x,y) \in \mathbb{R}^2 ; \underline{x^2+y^2 = 200}, \underline{x < 100}, \underline{y > 3}\}$ je borelovská

⌈ totiž je průnik ot. a uz. množin ⌋

• $\mathbb{Q}^2 = \{(x,y) \in \mathbb{R}^2 ; x \in \mathbb{Q}, y \in \mathbb{Q}\}$ je borelovská

⌈ $\bigcup_{\substack{k,l \in \mathbb{N} \\ l \neq 0 \\ p,q \in \mathbb{N} \\ q \neq 0}} \{ (\frac{\pm k}{l}, \frac{\pm p}{q}) \}$ je SPOČETNÉ SJEDNOCENÍ UZ. MNOŽIN $(\{q\}, q \in \mathbb{Q}^2)$

JEDNOPRVKOVÝCH. ⌋

• POZN.: • A SPOČETNÁ $= \exists f: A \to \mathbb{N}$ BIJEKCE

• $\mathbb{Q}$ SPOČETNÁ

• A, B SPOČETNÉ $\Rightarrow$ $A \times B$ SPOČETNÁ

• $(A_n)_{n \in \mathbb{N}}$ SPOČETNÁ $\Rightarrow \bigcup_{n \in \mathbb{N}} A_n$ SPOČETNÁ

TJ. $\mathbb{Q}^2 = \bigcup_{q \in \mathbb{Q}^2} \{q\}$ je borelovská

PLATÍ: $[0,1]$ je nespočetná

⌈ obecně: $(a,b) \subset \mathbb{R}$, $a < b$ není SPOČETNÝ ⌋

• $\{0\} \cup \{\frac{1}{n};\ n\in\mathbb{N}\} \subseteq \mathbb{R}$ je borelovská!

⊤ • je spočetná (sjednocení spočetných)

$\Rightarrow \; = \bigcup_{q\in\{0\}\cup\{\frac{1}{n};\ n\in\mathbb{N}\}} \{q\}$ • je spoč. sjednocení uz. = Borel.

• je dokonce uz. .... ⊥

• $[0,1) \times [2,3]$ je borelovská dle Lemmatu 3.2

NÁZNAK DŮKAZ LEMMATU 3.2:

OZNAČ • BÚNO $B = \mathbb{R}^m$

⊤ $A\times B = (A\times\mathbb{R}^m) \cap (\mathbb{R}^n\times B)$ ⊥

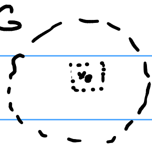
• POLOŽ $\mathcal{A} = \{A \subset \mathbb{R}^n;\ A\times\mathbb{R}^m$ je borel.$\}$

PAK • $\mathcal{A}$ je σ-algebra

• $\mathcal{A}$ obsahuje OT. MNOŽINY

(protože G ot. se dá napsat jako spoč. sjednocení OBDÉLNÍKŮ)

G



$\Rightarrow \mathcal{A} \supseteq \mathcal{B}(\mathbb{R}^n)$ „⊠"

POJEM MÍRY:

PŘ: a) $X=\{1,2,3,4,5,6\}$, $\mathcal{A} = \mathcal{P}(X)$,

$\mu(A)$ = POČET PRVKŮ A, $A \subseteq X$

⊤ • $\mu(\emptyset) = 0$ ✓ ; • $\mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu(A_n)$ ✓ disj. ⊥

b) $X=\{1,2,3,4,5,6\}$, $\mathcal{A} = \mathcal{P}(X)$

$$\mu(A) = \sum_{i \in A} \mu(\{i\}), \quad \text{kde} \quad \mu(\{i\}) = \frac{1}{6}.$$

SIMULUJE HÁZENÍ KOSTKOU (μ JE PRST)

$\mu(A)$ = PRST ŽE HODÍM ČÍSLO Z A

c) X libovolná, $\mathcal{A} = \mathcal{P}(X)$, $a \in X$ dáno

$$\delta_a(A) = \begin{cases} 0 & \dots\ a \notin A \\ 1 & \dots\ a \in A \end{cases} \quad \text{(DIRACOVA MÍRA)}$$

[DCV: OVĚŘIT ŽE JE MÍRA]

Pozn.: EXISTUJE μ MÍRA NA $\mathcal{B}(\mathbb{R}^n)$, $\mu \supseteq \ell^n$. PAK PRO NI PLATÍ:

- $\mu(\{1\}) = \mu\left(\bigcap_{n \in \mathbb{N}} (1-\frac{1}{n}, 1+\frac{1}{n})\right) = \lim_{n \to \infty} \mu(1-\frac{1}{n}, 1+\frac{1}{n})$
  $= \lim_{n\to\infty} \frac{2}{n} = \underline{\underline{0}}.$

  PODOBNĚ: $\mu(\{x\}) = 0$, $x \in \mathbb{R}^n$

  Tedy, $\mu(A) = 0$ pro A spočetnou

- $\mu([1,3] \setminus \{2\}) = \mu([1,3]) - \mu(\{2\}) = 2 - 0 = \underline{\underline{2}}.$

DK TRDITVE 3.3 :

(i) Naj $A \subseteq B$, potem
$$\mu(B) = \mu\big(A \cup (B\setminus A)\big) = \mu(A) + \mu(B\setminus A)$$

Torej • $\mu(A) \le \mu(A) + \mu(B\setminus A) = \mu(B)$ ✓

• $\mu(B\setminus A) < \infty \Rightarrow$
$$\mu(A) = \mu(B) - \mu(B\setminus A)$$

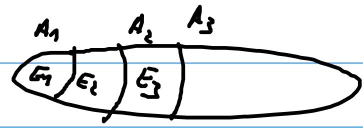
(ii) Naj $(A_n)_{n\in\mathbb{N}}$ jim z $\mathcal{A}$.

Položi
$$E_1 := A_1 \quad \in \mathcal{A}$$
$$E_2 := A_2 \setminus A_1 \quad \in \mathcal{A}$$
$$\vdots$$
$$E_{n+1} := A_{n+1} \setminus \bigcup_{i=1}^{n} A_i \quad \in \mathcal{A}, \quad n \in \mathbb{N}$$

Potem $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$ in E_n jim disj., torej
$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n) \overset{(i)}{\le} \sum_{n=1}^{\infty} \mu(A_n).$$

(iii) Naj $\underset{\in\mathcal{A}}{A_1} \subseteq \underset{\in\mathcal{A}}{A_2} \subseteq \underset{\in\mathcal{A}}{A_3} \subseteq \dots$

Ustvarimo množico $(E_n)_{n=1}^{\infty}$ kakor v Dk (ii)

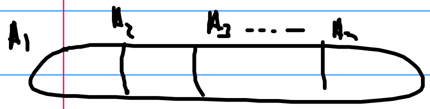


Potem
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n)$$
$$= \mu(A_1) + \sum_{n=1}^{\infty} \mu(A_{n+1}\setminus A_n) = \lim_{N\to\infty} \mu(A_1) + \sum_{n=1}^{N} \mu(A_{n+1}\setminus A_n)$$
$$= \lim_{N\to\infty} \mu\Big(A_1 \cup \bigcup_{n=1}^{N} A_{n+1}\setminus A_n\Big) = \lim_{N\to\infty} \mu(A_{N+1}).$$

(iv) Ať $\underset{\in\mathcal{A}}{A_1} \supseteq \underset{\in\mathcal{A}}{A_2} \supseteq \underset{\in\mathcal{A}}{A_3} \supseteq \dots$

Pak

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) \overset{(i)}{=} \mu(A_1) - \mu\Big(\underbrace{A_1 \setminus \bigcap_{n=1}^{\infty} A_n}_{\bigcup_{n=1}^{\infty} A_1\setminus A_n}\Big) = \mu(A_1) - \mu\left(\bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\right)$$



$$\overset{(iii)}{=} \mu(A_1) - \lim_{n\to\infty} \overbrace{\mu(A_1 \setminus A_n)}^{= \mu(A_1) - \mu(A_n)}$$

$$\overset{AL}{=} \lim_{n\to\infty} \mu(A_n) . \quad \boxtimes$$

→ **POZNÁMKA O NEBOREL. MNOŽINÁCH**:

$V \subseteq [0,1]$ def. takto:

• $r \in \mathbb{R}$ ... označ $r + \mathbb{Q} = \{r+q;\ q \in \mathbb{Q}\}$ ... pak $\mathbb{R} = \bigcup_{r\in\mathbb{R}} (r+\mathbb{Q})$

Z každé množiny $(r+\mathbb{Q})$ vyberu jeden prvek $v \in [0,1]$

[V má vlastnost: $\forall r \in \mathbb{R}\ \exists!\, v \in V:\ r - v \in \mathbb{Q}$]

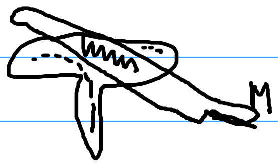
Pak V není borelovská (důkaz ...)

• Př.: • $X = \{1,2,3,4,5,6\}$, $\mathcal{A} = \mathcal{P}(X)$, $\mu(\{i\}) = \frac{1}{6}$.

Pak $\mu(\{1,2,3\}) = \mu(\{1\} \cup \{2\} \cup \{3\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \underline{\underline{\frac{1}{2}}}$.

• $(\delta_a + 4\delta_b)(\{a,b\}) = \delta_a(\{a,b\}) + 4\,\delta_b(\{a,b\})$
$= 1 + 4 = \underline{\underline{5}}$.

• obr k definici μ^*-měř. množiny (dle Carathéodoryho):



Př: $\bullet\ \lambda(\{1\}) = \lambda\left(\bigcap_{n\in\mathbb{N}} \left(1-\frac{1}{n},\ 1+\frac{1}{n}\right)\right) \overset{T\,3.2}{=} \lim_{n\to\infty} \lambda\left(\left(1-\frac{1}{n},\ 1+\frac{1}{n}\right)\right)$

$$= \lim_{n\to\infty} \frac{2}{n} = \underline{\underline{0}}.$$

(PODOBNĚ: $\lambda^n(\{x\}) = 0$ pro $x \in \mathbb{R}^n$)

$\bullet\ \lambda(\mathbb{N}) = 0$ $\ulcorner\ \lambda(\mathbb{N}) = \lambda\left(\bigcup_{n\in\mathbb{N}} \{n\}\right) = \sum_{n=1}^{\infty} \lambda(\{n\}) = \sum_{n=1}^{\infty} 0 = \underline{\underline{0}}\ \lrcorner$

(PODOBNĚ: $\lambda^n(A) = 0$ pro $A \subset \mathbb{R}^n$ spočetnou)

$\bullet\ \lambda\left((2,8) \setminus \{3,4,5\}\right) = \lambda((2,8)) - \lambda(\{3,4,5\})$

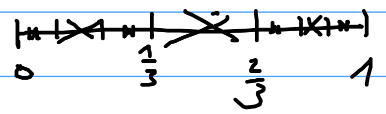
$$= 6 - 0 = \underline{\underline{6}}.$$

$\bullet\ \lambda\left(\{a_n\}_{n=1}^{\infty}\right) = 0$ pro $\{a_n\} \subset \mathbb{R}$ posloupnost

$\ulcorner$ viz. $\lrcorner$

$\bullet\ \lambda$ (Cantorovo diskontinuum)

$\ulcorner$



... $C_0 = [0,1]$

C_1 ... 2 intervaly délky $\frac{1}{3}$

$\vdots$

C_{n+1} ... 2^{n+1} intervalů (disjunktních) délky $\frac{1}{3^{n+1}}$

$C = \bigcap C_n$

Tedy $\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} 2^{n+1} \cdot \frac{1}{3^{n+1}} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n+1} = \underline{\underline{0}}$ $\lrcorner$

Dk v 3.6: DŮKAZ PRO $f: D \to \mathbb{R}$ MĚŘITELNOU.

Označ $\tilde{\mathcal{A}} = \{E \subseteq \mathbb{R};\ \{f \in E\} \in \mathcal{A}\}$

• PAK $\tilde{\mathcal{A}}$ je σ- algebra

⌐ • $\emptyset \in \tilde{\mathcal{A}}$ protože $\{x;\ f(x) \in \emptyset\} = \emptyset \in \mathcal{A}$

• $\mathbb{R} \in \tilde{\mathcal{A}}$ protože $\{x;\ f(x) \in \mathbb{R}\} = D \in \mathcal{A}$

• $E \in \tilde{\mathcal{A}}$, pak $\{x;\ f(x) \in \mathbb{R} \setminus E\} = X \setminus \underbrace{\{x;\ f(x) \in E\}}_{\in \mathcal{A}} \in \mathcal{A}$

tedy $\mathbb{R} \setminus E \in \tilde{\mathcal{A}}$

• Ať $(E_n)_{n=1}^{\infty}$ je posl. z $\tilde{\mathcal{A}}$, pak

$$\{x;\ f(x) \in \bigcup E_n\} = \bigcup_{n=1}^{\infty} \underbrace{\{x;\ f(x) \in E_n\}}_{\in \mathcal{A}} \in \mathcal{A}$$

tedy $\bigcup E_n \in \tilde{\mathcal{A}}$.

⌟

• $\tilde{\mathcal{A}}$ obsahuje otevřené množiny

⌐ • $\tilde{\mathcal{A}} \supseteq$ ot. intervaly dle definice měřitelnosti.

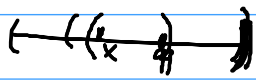
• Pak ale obsahuje i ot. množiny, protože každá ot. množina je sjednocením (SPOČETNÝM) ot. intervalů

⌐ Ať $G \subseteq \mathbb{R}$ ot. → Buď $x \in G$ ... $\exists R > 0:\ B(x,R) \subseteq G$

$B(x,R) = (x-R, x+R)$

NEPOVINNÁ ČÁST

$\Rightarrow \exists q \in (x - \frac{R}{2}, x + \frac{R}{2}) \in G$



Pak $x \in (q - \frac{R}{2},\ q + \frac{R}{2}) \subseteq G$

Celkem: $G = \bigcup_{\substack{q \in G \cap \mathbb{Q} \\ R \in \mathbb{Q} \text{ tž } (q-R, q+R) \subseteq G}} (q - R, q + R)$ ⌟

$\Rightarrow \mathcal{B}(\mathbb{R}) \subset \tilde{\mathcal{A}}$

☒

Dk tvrzení 3.8: (i), (ii) vynech

~~(Lemma)~~ Fakt: $f: D \to \mathbb{R}$ je měř. $\iff$ $\{f > \alpha\} \in \mathcal{A}$ pro $\alpha \in \mathbb{R}$

Dk.: $(a,b) = (a,\infty) \cap (-\infty, b)$, tedy

• $\{f \in (a,b)\} = \underbrace{\{f > a\}}_{\in \mathcal{A}} \cap \underbrace{\{f < b\}}_{\in \mathcal{A}} \in \mathcal{A}$ viz.

$\{f < b\} = \{f \geq b\}^c = \Big(\bigcap \overbrace{\{f > b - \tfrac{1}{n}\}}^{\in \mathcal{A}}\Big)^c \in \mathcal{A}$

$f(x) \geq b \iff f(x) > b - \frac{1}{n}, \ n \in \mathbb{N}$ □

(iii) $\{\chi_A > \alpha\} = \{x;\ \chi_A(x) > \alpha\} =$
- pokud: $\alpha \geq 1$, pak $= \emptyset \in \mathcal{A}$
- pokud: $\alpha < 0$, pak $= X \in \mathcal{A}$

Pokud: $\alpha \in [0,1)$:

$= \{x;\ \chi_A(x) > \alpha\} = \{x;\ \chi_A(x) = 1\} = \{x;\ x \in A\} = A \in \mathcal{A}$

(iv) – (v) vynech ☒

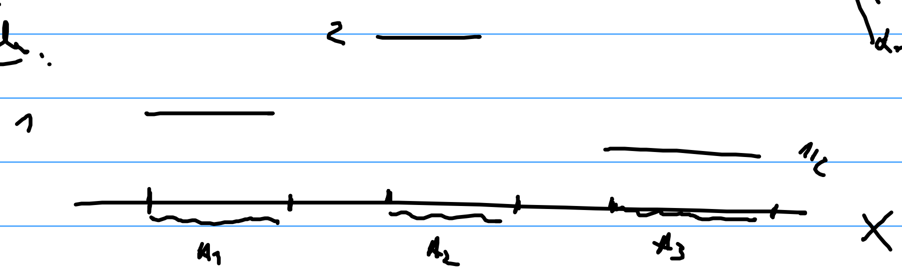
Př.:

• f má konečně mnoho hodnot, pak $f^{-1}(\{i\}) \in \mathcal{A}$ pro $i \in \operatorname{Rg} f$

$\implies f$ je měřitelná

Tj. $f(x) = \begin{cases} \alpha_1 & \dots\ x \in A_1 \\ \alpha_2 & \dots\ x \in A_2 \\ \vdots \\ \alpha_n & \dots\ x \in A_n \end{cases} \in \mathcal{A}$

Obr.:



$\implies f = \chi_{A_1} + 2\chi_{A_2} + \frac{1}{2}\chi_{A_3}$,

obecně $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, tedy je ~~...~~ měř.

- $f(x,y) = \operatorname{arctg}(x+y)$ .... je SPOJ. $\Rightarrow$ BORELOVSKY MĚŘ.

- $$f(x,y) = \begin{cases} \operatorname{arctg}(x+y) & \dots \ (x,y) \in B((1,1),3) \\ \exp(x+y) & \dots \ (x,y) \in B((50,50),4) \\ 1 & \dots \ \text{jinak} \end{cases}$$

je měř., neboť

$$f = \chi_{B((1,1),3)} \cdot \operatorname{arctg}(x+y) + \chi_{B((50,50),4)} \cdot \exp(x+y)$$

$$+ \chi_{\mathbb{R}^2 \setminus \left(B((1,1),3) \cup B((50,50),4)\right)}$$

$\Rightarrow$ je borel. měř.

- $$f(x) = \begin{cases} 1-x^2 & \dots \ x \in [-1,1] \setminus \mathbb{Q} \\ 0 & \dots \ \text{jinak} \end{cases}$$ je měřitelná (borelovská), neboť

$$f(x) = \chi_{[-1,1]\setminus \mathbb{Q}}(x) \cdot (1-x^2)$$

LEBESGUEŮV INTEGRÁL

TEST V MODLU:

$f: \mathbb{R} \to \mathbb{R}$

- $f = 0$ s.v. $\Rightarrow$ f je Leb. měř.

Γ $\{f > a\} = \left(\{f > a\} \cap \{f \neq 0\}\right) \cup \left(\{f > a\} \cap \{f = 0\}\right)$

$= \underbrace{\left(\{f > a\} \cap \{f \neq 0\}\right)}_{\subseteq \{f \neq 0\} \in \mathcal{M}(\lambda)}^{\in \mathcal{M}(\lambda)} \cup \underbrace{\{f = 0\}}_{\in \mathcal{M}(\lambda)}$ ... POKUD $0 > a$

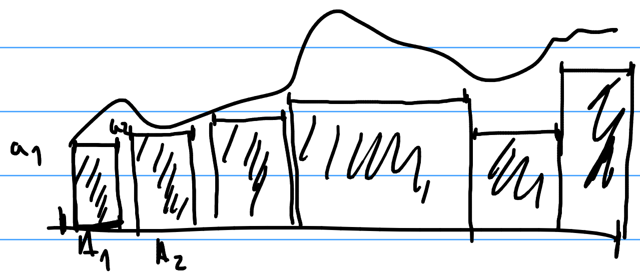
$\underbrace{\text{——}\,\|\,\text{——}}_{\in \mathcal{M}(\lambda)} \cup \overset{\in \mathcal{M}(\lambda)}{\emptyset}$ ... POKUD $0 \leq a$

$\Rightarrow$ a vyjádření měř. množin je měř. $\lrcorner$

- **PRO $f \geq 0$:**

$$\int_X f \, d\mu = \sup \left\{ \sum_{i=1}^{n} a_i \, \mu(A_i) ;\ 0 \leq a_i \leq f \text{ na } A_i,\ A_1 \cup \dots \cup A_n = X,\ A_i \cap A_j = \emptyset \right\}$$

Obr.:



FAKT (0): $A \in \mathcal{A} \Rightarrow \int_X \chi_A(x) \, d\mu(x) = \mu(A)$

Dk.: „$\leq$": zvol rozklad $(A_i)_{i=1}^n$ množiny X

a $0 \leq a_i \leq \chi_A(x)$, $i = 1, \dots, n$.

pro $x \in A_i$.

Speciálně, pokud $\exists\, x \in A_i \setminus A$, pak $a_i = 0$

Tedy: $a_i = 0$ pokud $A_i \not\subseteq A$

$$\Rightarrow \underbrace{\sum_{i=1}^{n} a_i \, \mu(A_i)} = \sum_{\substack{i=1 \\ A_i \subseteq A}}^{n} a_i \, \mu(A_i) \overset{a_i \leq 1}{\leq} \sum \mu(A_i)$$

$$= \mu\left(\bigcup A_i\right) \leq \underline{\mu(A)},$$

„$\geq$"

Zvolme $A_1 := A$, $A_2 := X \setminus A$

$a_1 = 1 \qquad a_2 = 0$

Pak

$$\underbrace{a_1}_{=1} \mu(A_1) + \underbrace{a_2}_{0} \mu(A_2) = \mu(A)$$

$\boxtimes$

$\Rightarrow$ DOKÁZALI JSME TÍM TVRZENÍ 3.9 (i)

DŮKAZ TVRZENÍ 3.9:

(ii): $\int_E f = 0 \Leftarrow \mu(E) = 0 \vee f = 0$ s. v.

[N. PŘÍPAD $f \geq 0$ (BÚNO) $\rightarrow$ APLIKUJI PAK NA f^+ a NA f^-

• Ať $\mu(E) = 0$, pak pro $\forall$ rozklad $(A_j)_{j=1}^{n}$ množiny X,

$\forall (a_j)$: $0 \leq a_j \leq f \cdot \chi_{E \cap D(f)}$

pak $A_j \cap (E \cap D(f))^c \Rightarrow a_j = 0$

Tedy $\int_E f \, d\mu = \int_X f \cdot \chi_{E \cap D(f)} \, d\mu = \sup \Big\{ \sum_{A_j \subseteq E} a_j \, \mu(A_j);$

$\ldots\ldots\ldots \Big\} = \sup \{0, \ldots\} = \underline{0}.$

$0 \leq \mu(A_j) \leq \mu(E) = 0$

• Ať $f = 0$ s. v. BÚNO: $f \geq 0$ [PAK APLIKUJI NA f^+ a f^-]

BÚNO: $E = X$, $D(f) = X$

[Protože pak aplikuji na $\left(\int_E f = \int_X \boxed{f \cdot \chi_{E \cap D(f)}}\right)$

$f \cdot \chi_{E \cap D(f)}$]

Pak $\int_X f \, d\mu = \sup \Big\{ \sum a_j \, \mu(A_j);\ 0 \leq a_j \leq f$ na A_j, (A_j) rozklad... $\Big\}$

$$= \sup \Big\{ \sum_{A_j \subseteq \{f \neq 0\}} a_j \, \mu(A_j) \; ; \; \dots \Big\} = \sup \{0, \dots\} = \underline{0}.$$

$A_j \cap \{f = 0\} \neq \emptyset \Rightarrow a_j = 0$ $\qquad$ $0 \leq \mu(A_j) \leq \mu(\{f \neq 0\}) = 0$

┘

(iii) $\int |f| < \infty \Rightarrow |f| < \infty$ s.v.

┌ ~~Důkaz~~ DK UMÍME ... ┘

~~(ADM)~~ BEZ DŮKAZU: (v), (vi)

(iv) BÚNO: $f \geq 0$ (APLIKUJEME PAK NA f^+ a f^-)

$$\Rightarrow \int_E f \, d\mu = \int_X f \cdot \chi_{E \cap D(f)} \, d\mu = \sup \Big\{ \sum_{A_j \subseteq E \cap D(f)} a_j \, \mu(A_j) \; ; \; \dots \Big\}$$

$A_j \cap (E \cap D(f))^c \neq \emptyset \Rightarrow a_j = 0$

$$\leq \mu(E) \cdot \sup_{x \in E \cap D(f)} |f(x)|.$$

$A_j \subseteq E \cap D(f) \subseteq E$ $\qquad$ $a_j \leq \chi_{E \cap D(f)} \cdot f \leq \sup_{x \in E \cap D(f)} |f(x)|$

┘

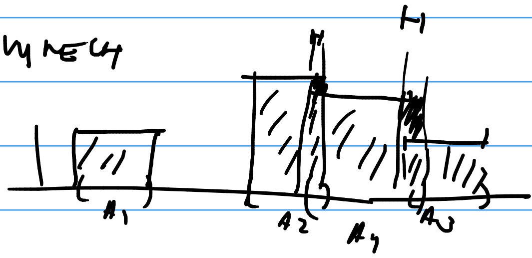
(vii) $\int f \, d\mu \in \mathbb{R} \iff \int |f| \, d\mu \in \mathbb{R}$

┌ $\int f \, d\mu \in \mathbb{R} \iff \underbrace{\int f^+ \, d\mu \in \mathbb{R} \;\; \& \;\; \int f^- \, d\mu \in \mathbb{R}}$

$$\int |f| \, d\mu \underset{(vi) + (f^+ + f^- = |f|)}{=} \int f^+ \, d\mu + \int f^- \, d\mu \in \mathbb{R} \iff$$

┘

(viii)+(ix): DK UMÍME



PŘ: ① A_i měř.

$$\int_X \sum_{i=1}^n a_i \, \chi_{A_i}(x) \, d\mu(x) \underset{T3.9\,(vi)}{=} \sum_{i=1}^n a_i \int_X \chi_{A_i}(x) \, d\mu \underset{T3.9\,(i)}{=} \sum_{i=1}^n a_i \, \mu(A_i)$$

PŘÍKL: $X = \{1,2\}$, $\mathcal{A} = \mathcal{P}(X)$, $f(i) = i+3$, $\mu(\{1\}) = \frac{1}{2} = \mu(\{2\})$

$$\int_X f \, d\mu = \int_{\{1\}} f \, d\mu + \int_{\{2\}} f \, d\mu = \int_{\{1\}} (1+3) \cdot \chi_{\{1\}} \, d\mu + \int_{\{2\}} (2+3)\, \chi_{\{2\}} \, d\mu =$$

$$= 4\, \mu(\{1\}) + 5\, \mu(\{2\}) = \frac{9}{2}.$$

[OBECNĚJI: $\int_X f \, d\mu = \sum_{i=1,2} f(i)\, \mu(\{i\})$ = „VÁŽENÝ PRŮMĚR f"]

- PRVNÍ INTERPRETACE INTEGRÁLU
- DRUHÁ INTERPRETACE: „POJEM OBSAHU PODGRAFU"
- TŘETÍ INTERPRETACE: $\int_{-\infty}^{\infty} f \, d\lambda$ ... $\int_A f \, d\lambda$ = „PRST ŽE $f \in A$" kde $f: \mathbb{R} \to \mathbb{R}$

PŘ:

- $(L)\text{-}\int_{(0,1)\setminus\{\frac{1}{2}\}} x \, d\lambda(x) = (L)\text{-}\int_{(0,1)} x \, d\lambda(x) = (R)\text{-}\int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

  $\lambda(\{\frac{1}{2}\}) = 0$, tedy $\int_{\{\frac{1}{2}\}} x \, d\lambda(x) = 0$

- $(L)\text{-}\int_{\mathbb{R}} \chi_{\mathbb{Q}}(x) \, d\lambda(x) = 0$

  $\lambda(\mathbb{Q}) = 0$, tj. $\chi_{\mathbb{Q}} = 0$ s.v.

POZN: $(R)\text{-}\int_0^1 \chi_{\mathbb{Q}}(x) \, dx$ neex. !

- $(L)\text{-}\int_{(-\frac{\pi}{2}, \frac{\pi}{2})} \cos x \, d\lambda = (L)\text{-}\int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \cos x \, d\lambda = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = \ldots = 2.$

  $\lambda((-\frac{\pi}{2}, \frac{\pi}{2})) = \lambda([-\frac{\pi}{2}, \frac{\pi}{2}])$,

  neboť: $\lambda(\{-\frac{\pi}{2}, \frac{\pi}{2}\}) = 0$

Př 1:

$$\overset{\text{V3 s}(\dots)+(2)}{\downarrow}$$

$$(L)\text{-}\int_{[0,\infty)} \frac{1}{x}\, d\lambda(x) = (L)\text{-}\int_{(0,\infty)} \frac{1}{x}\, d\lambda(x) = (N)\text{-}\int_0^\infty \frac{1}{x}\, dx$$

$$= \left[\log x\right]_0^\infty = \infty - (-\infty) = \underline{\underline{+\infty}}.$$

Př 2: $(L)\text{-}\int_{(1,\infty)} \frac{\sin x}{x}\, d\lambda(x)$ neex.

$\ulcorner$ $(L)\text{-}\int_{(1,\infty)} \left|\frac{\sin x}{x}\right| d\lambda(x) = (N)\text{-}\int_1^\infty \left|\frac{\sin x}{x}\right| dx = +\infty$

$\Rightarrow (L)\text{-}\int \frac{\sin x}{x}\, d\lambda(x) \notin \mathbb{R}$

• pro spor, ať existuje. [tj. má smysl $\int \left(\frac{\sin x}{x}\right)^+ - \int \left(\frac{\sin x}{x}\right)^-$]

Pak

pokud $(L)\text{-}\int_1^\infty \left(\frac{\sin x}{x}\right)^+ = +\infty$ & $(L)\text{-}\int_1^\infty \left(\frac{\sin x}{x}\right)^- \in \mathbb{R}$

$\quad\quad \| \quad\quad\quad\quad\quad\quad\quad\quad \|$

$(N)\text{-}\int$ −"− $\quad\quad\quad$ (N) −"−

tedy pak $(N)\text{-}\int_1^\infty \frac{\sin x}{x}\, dx = +\infty$ ↯ spor

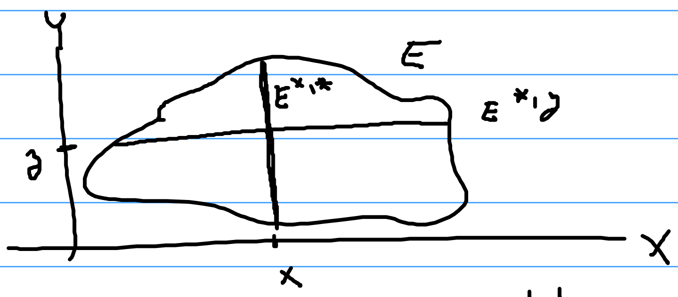
Analogicky vede na spor případ

$(L)\text{-}\int_1^\infty \left(\frac{\sin x}{x}\right)^+ \in \mathbb{R}$ & $(L)\text{-}\int \left(\frac{\sin x}{x}\right)^- = +\infty$

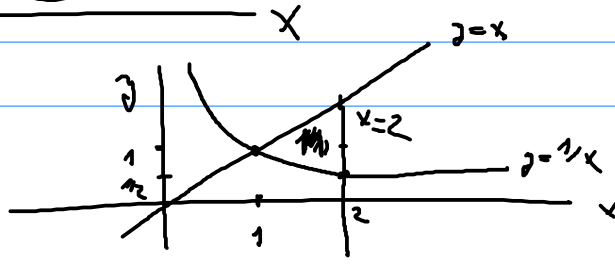
$\Rightarrow (L)\text{-}\int_1^\infty \left(\frac{\sin x}{x}\right)^+ = \infty = (L)\text{-}\int_1^\infty \left(\frac{\sin x}{x}\right)^- \Rightarrow$ integrál neex. $\lrcorner$

FUBINIOVA VĚTA)

ŘEZY:



PŘ V.1.a: dk.:



Tedy $M = \{ (x,y);\ x \in [1,2],\ \frac{1}{x} \le y \le x \}$

je měřitelná (je uz. $\Rightarrow$ tedy borel.)

Počítejme:

$$\lambda^2(M) = \int_{\mathbb{R}^2} \chi_M(x)\, d\lambda^2(x) = \int_M 1\, d\lambda^2(x)$$

$$\underset{\substack{\downarrow \\ \text{FUBINI (} 1 \ge 0 \text{, TEDY PŘEDP. SPLNĚNY)}}}{=} \int_1^2 \int_{1/x}^{x} 1 \, dy \, dx$$

$$= \int_1^2 x - \frac{1}{x}\, dx = \left[\frac{x^2}{2} - \lg x \right]_1^2$$

$$= \dots = \underline{\underline{\frac{3}{2} - \lg 2}}.$$

2. zápis:

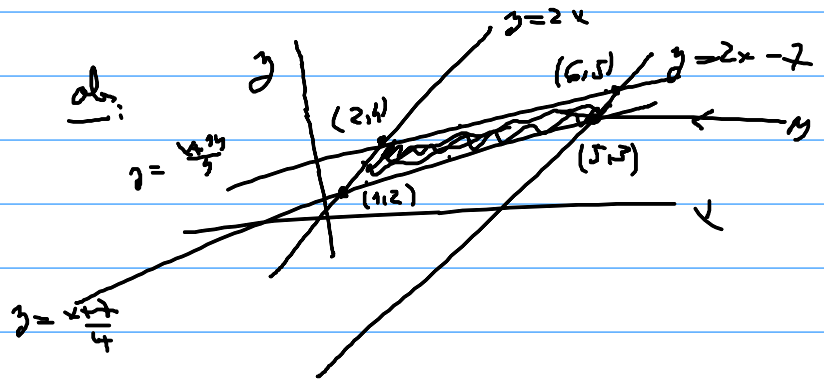
$$\lambda^2(M) = \int_M 1\, d\lambda^2 \overset{\text{FUBINI}}{=} \int_{1/2}^{1} \int_{1/y}^{2} 1 \, dx \, dy$$

$$+ \int_1^2 \int_y^2 1\, dx\, dy$$

$$= \dots = \underline{\underline{3/2 - \lg 2}}.$$

PŘ

ob.:



$$M = \{ (x,y) \in \mathbb{R}^2;\ x \in [1,2],\ \frac{x+7}{4} \le y \le 2x \}$$

$$\cup \{ (x,y) \in \mathbb{R}^2;\ x \in [2,5],\ \frac{x+7}{4} \le y \le \frac{x+14}{5} \}$$

$$\cup \{ (x,y) \in \mathbb{R}^2;\ x \in [5,6],\ 2x-7 \le y \le \frac{x+14}{5} \}$$

je měřitelná (je uz. $\Rightarrow$ je borel.)

A

$$\lambda^2(M) = \int_M 1 \, d\lambda^2(x,y) \overset{\text{FUBINI}}{=} \int_1^2 \int_{\frac{x+7}{4}}^{2x} 1 \, dy \, dx$$

$$+ \int_2^5 \int_{\frac{x+7}{4}}^{\frac{x+14}{4}} 1 \, dy \, dx$$

$$+ \int_5^6 \int_{2x-7}^{\frac{x+14}{4}} 1 \, dy \, dx$$

$$= \int_1^2 \left(2x - \frac{x+7}{4}\right) dx + \dots$$

$$= \dots = \underline{\underline{7}}.$$

PŘ

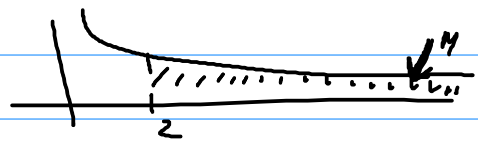
$$M = \{(x,y) ; \ x > 2, \ 0 < y < 1/x\}$$

je měř. (je OT. $\Rightarrow$ borel.)

Máme
$$\lambda^2(M) = \int_M 1 \, d\lambda^2(x,y) = \int_2^\infty \int_0^{1/x} 1 \, dy \, dx$$

$$= \int_2^\infty \frac{1}{x} \, dx = [\log x]_2^\infty = \underline{\underline{\infty}}.$$

obr.:



POZN.: VŽDY PLATÍ: $\lambda^2(M) = \int_M 1 \, d\lambda^2(x,y)$ (pokud M je měř.)

PŘ

$$\int_{[3,4]\times[1,2]} \frac{1}{(x+y)^2} \, d\lambda^2(x,y) =$$

$$\overset{\text{FUBINI } \left[\frac{1}{(x+y)^2} \geq 0\right]}{=} \int_3^4 \int_1^2 \frac{1}{(x+y)^2} \, dy \, dx$$

$$= \int_3^4 \left[-\frac{1}{(x+y)} \right]_{y=1}^2 dx$$

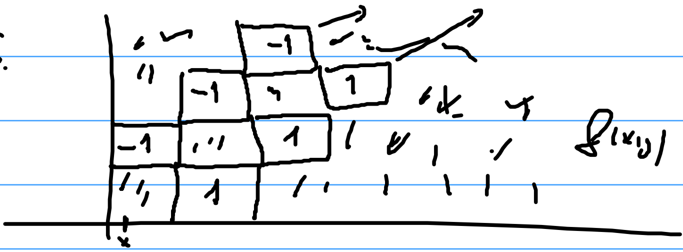
$$= \int_3^4 -\frac{1}{x+2} + \frac{1}{x+1} \, dx = \left[\log\left(\frac{x+1}{x+2}\right) \right]_3^4$$

$$= \lg\left(\frac{5}{6}\right) - \lg\left(\frac{4}{5}\right) = \lg\left(\frac{5}{6}\cdot\frac{5}{4}\right) = \lg\left(\frac{25}{24}\right)$$

2. ZPŮSOB: $$\int_1^2 \int_3^4 \frac{1}{(x+y)^2}\, dx\, dy = \dots = \lg\left(\frac{25}{24}\right)$$

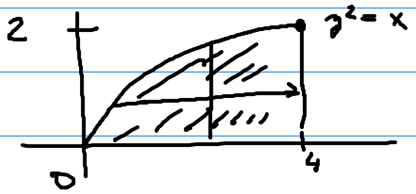
UPOZORNĚNÍ! FUBINIOVA VĚTA PRO OBECNÉ MĚŘ. MNOŽINY A MĚŘ. FCE NEPLATÍ!

PŘ.:



Pak $$\int_0^\infty \int_0^\infty f(x,y)\, dy\, dx = -1$$
$$\neq$$
$$\int_0^\infty \int_0^\infty f(x,y)\, dx\, dy = 1$$

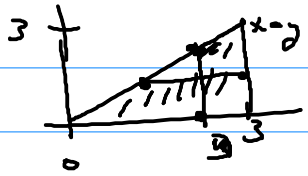
PŘ. $$\int_0^2 \int_{y^2}^4 1\, dx\, dy = \int_0^2 (4-y^2)\, dy = \left[4y - \frac{y^3}{3}\right]_0^2$$
$$= 8 - \frac{8}{3} = \frac{16}{3}.$$



Obráceně:
$$\int_0^4 \int_0^{\sqrt{x}} 1\, dy\, dx = \int_0^4 \sqrt{x}\, dx = \left[\sqrt{x^3}\,\frac{2}{3}\right]_0^4$$
$$= \frac{4.2.2}{3} = \frac{16}{3}.$$

PŘ: $$\int_0^3 \int_y^3 e^{x^2}\, dx\, dy =$$

$$\overset{\text{FUBINI } (e^{x^2} \geq 0)}{=} \int_0^3 \int_0^x e^{x^2}\, dy\, dx = \int_0^3 x e^{x^2}\, dx$$



$$= \left| \begin{array}{l} t = x^2 \\ dt = 2x \end{array} \right| = \int_0^9 \frac{1}{2} e^t\, dt = \frac{1}{2} \left[e^t \right]_0^9 = \underline{\frac{1}{2}(e^9 - 1)}.$$

Př.

$$M = \{ (x,y,z) \in \mathbb{R}^3 ;\ 0 \leq x \leq 1,\ 0 \leq y \leq x,\ 0 \leq z \leq e^{-x^2} \}$$

je měř. (je uz. $\Rightarrow$ borel)

Máme

$$\lambda^3(M) = \int_M 1\, d\lambda^3(x,y,z) \overset{\text{FUBINI}}{=} \int_0^1 \int_0^x \int_0^{e^{-x^2}} 1\, dz\, dy\, dx$$

$$= \int_0^1 x e^{-x^2}\, dx = -\frac{1}{2} \left[e^{-x^2} \right]_0^1 = \underline{-\frac{1}{2}\left(\frac{1}{e} - 1\right)}.$$

Př.

$$\bullet\ 6x^2 - 2xy \underset{y \leq 3x - x^2}{\geq} 6x^2 - 2x(3x - x^2) = 2x^3 \overset{x \in [0,2]}{\geq} 0$$

$$\Rightarrow M = \{ (x,y,z) \in \mathbb{R}^3 ;\ 0 \leq z \leq 6x^2 - 2xy,\ x \in [0,2],\ y \in [x, 3x - x^2] \}$$

je měř. (protože je uz. $\Rightarrow$ borel.)

A máme

$$\lambda^3(M) = \int_M 1\, d\lambda^3(x,y,z) \overset{\text{FUBINI } (1 \geq 0)}{=} \int_0^2 \int_x^{3x-x^2} \int_0^{6x^2-2xy} 1\, dz\, dy\, dx$$

$$= \int_0^2 \int_x^{3x-x^2} (6x^2 - 2xy)\, dy\, dx$$

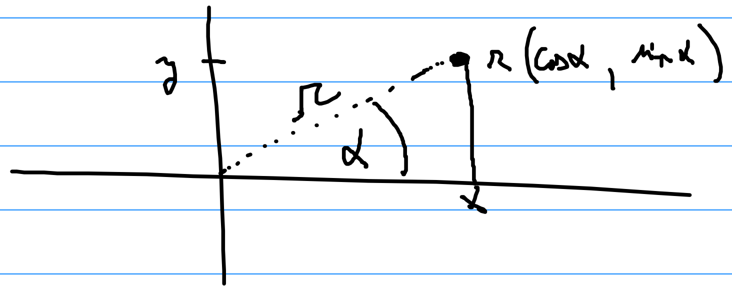
$$= \int_0^2 \left[6x^2 y - 2x \frac{y^2}{2} \right]_{y=x}^{3x-x^2} dx$$

$$= \int_0^2 \left(6x^2(3x-x^2) - x(3x-x^2)^2\right) - \left(6x^3 - x^3\right) dx$$

$$= \dots = \underline{\underline{\frac{16}{3}}}.$$

SUBSTITUCE VE 2D

POLÁRNÍ SOUŘADNICE :



DŮKAZ VĚTY 3.15 PRO $a = 1 = b$:

$\varphi(r,\alpha) := (r\cos\alpha,\ r\sin\alpha)$, $G := \{(r,\alpha);\ r>0,\ \alpha\in(-\pi,\pi)\}$

- $\varphi \in \mathcal{C}^1(G)$ zřejmé ; OZNAČ $x = r\cos\alpha$, $y = r\sin\alpha$
- φ je PROSTÁ :
  - → $r = \sqrt{x^2+y^2}$ ... tedy (x,y) jednoznačně určí r
  - → $\alpha = 2\,\mathrm{arctg}\left(\dfrac{y}{x+\sqrt{x^2+y^2}}\right)$ .... tedy (x,y) jednoznačně určí α

$$\frac{y}{x+\sqrt{x^2+y^2}} = \frac{r\sin\alpha}{r\cos\alpha + r} = \frac{\sin\alpha}{\cos\alpha+1} =$$

$s(2x)=2sc$
$c(2x)=c^2-s^2$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}\ \underbrace{-\sin^2\frac{\alpha}{2}+1}_{=+\cos^2\frac{\alpha}{2}}} = \mathrm{tg}\,\frac{\alpha}{2}$$

... zbývá APLIKOVAT arctg na obě strany ROVNOSTI

$\Rightarrow \varphi$ je prostá

- $|J_\varphi(r,\alpha)| = \left|\begin{pmatrix}\cos\alpha & -r\sin\alpha\\ \sin\alpha & r\cos\alpha\end{pmatrix}\right| = r\cos^2\alpha + r\sin^2\alpha = r > 0$

$\Rightarrow \varphi$ je REGULÁRNÍ

• $\lambda^2(\varphi(G)^c) = 0$: STAČÍ ~~TVRZENÍ~~ : $\varphi(G) \supseteq \left(\mathbb{R}^2 \setminus (-\infty,0]\times\{0\}\right)$

Důkaz: Zvol $(x,y) \in \mathbb{R}^2 \setminus (-\infty,0]\times\{0\}$ ....

POLOŽ $r = \sqrt{x^2+y^2}$, $\alpha = 2\,\mathrm{arctg}\left(\frac{y}{\underbrace{x+\sqrt{x^2+y^2}}_{\neq 0}}\right)$

$\neq 0$ - L.F. $\sqrt{x^2+y^2} = -x$

$x^2+y^2 = x^2$

$y^2 = 0$

$y = 0$

$\Rightarrow$ POKUD $x+\sqrt{x^2+y^2} = 0$, PAK $y=0$ a $x+\sqrt{x^2} = 0$

$\Rightarrow (x,y) \in (-\infty,0]\times\{0\}$ $\underline{\text{SPOR}}$

PAK $(x,y) = (r\cos\alpha, r\sin\alpha)$ ∎

Tedy $\varphi(G)^c \subseteq \underbrace{(-\infty,0]\times\{0\}}_{\lambda^2\text{-míra NULA}} \Rightarrow \lambda^2(\varphi(G)^c) = 0$.

$\Rightarrow$ APLIKUJEME 3.14 :

$\varphi^{-1}(E) = \varphi^{-1}(E\cap\varphi(G))$

$$\int_{E\cap\varphi(G)} f\,d\lambda^2 \overset{3.14}{=} \int_{\varphi^{-1}(E)} f(r\cos\alpha, r\sin\alpha)\cdot r\,d\lambda^2$$

$$\| \quad \int_E f\,d\lambda^2 \quad \leftarrow \lambda^2(E\setminus\varphi(G)) = 0$$

PŘÍKLADY:

$$: \int_{\underbrace{\{x^2+y^2\le 1\}}_{M}} \overbrace{\frac{1}{\sqrt{1-x^2-y^2}}}^{f}\,d\lambda^2$$

• M je měř., protože je uz. (a tedy borelovská!)

• f je SPOJ. na $\{x^2+y^2<1\}$ $\xrightarrow{\text{s.v. na } M}$ $\Rightarrow$ je měř.

POLÁRNÍ SOUŘ. (V 3.15)

$$\int_M f(x,y)\, d\lambda^2 = \left| \begin{matrix} x = r\cos\alpha \\ y = r\sin\alpha \end{matrix}\right| = \int_{\{(r,\alpha)\in G;\ \underbrace{(r\cos\alpha)^2+(r\sin\alpha)^2}_{=r^2}\le 1\}} \frac{r}{\sqrt{1-\frac{r^2}{}}}\, d\lambda^2(r,\alpha)$$

$$\overset{\text{FUBINI}}{=} \int_{-\pi}^{\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr\, d\alpha \overset{\text{FUBINI}}{=} 2\pi\int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr$$

$$\int_0^1\int_{-\pi}^{\pi} \frac{r}{\sqrt{1-r^2}}\, d\alpha\, dr = \int_0^1 \frac{r}{\sqrt{1-r^2}} \int_{-\pi}^{\pi} d\alpha\, dr = 2\pi\int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr$$

$$= \left|\begin{matrix} t=r^2\\ dt=2r\end{matrix}\right| = 2\pi \int_0^1 \frac12 \frac{1}{\sqrt{1-t}}\, dt = 2\pi\left[-\sqrt{1-t}\right]_0^1$$

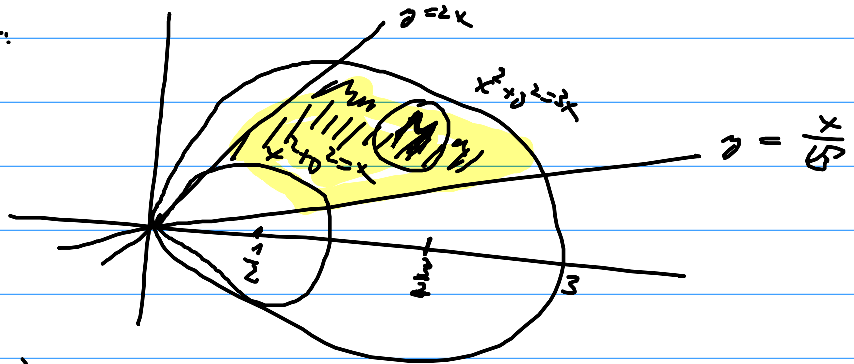
$$= \underline{\underline{2\pi}}.$$

Př

$$M = \left\{ \frac{x}{\sqrt3}\le y\le 2x,\quad x\le x^2+y^2\le 3x \right\}$$

$$f(x,y) = \frac{1}{(x^2+y^2)^2}$$

Ř.:



- $x^2+y^2\le 3x \iff (x-\frac32)^2+y^2\le\frac94$
- $x^2+y^2\ge x \iff (x-\frac12)^2+y^2\ge\frac14$

PŘEVOD DO POLÁRNÍCH SOUŘADNIC:

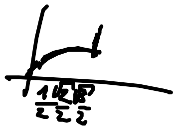
- $\frac{r\cos\alpha}{\sqrt3}\le r\sin\alpha\le 2r\cos\alpha$
- Protože $(x,y)\in M \Rightarrow x>0,\ y>0 \Rightarrow \alpha\in(0,\frac{\pi}{2})$

$\Rightarrow \operatorname{tg}\alpha\in\left[\frac{1}{\sqrt3},2\right] \Rightarrow \alpha\in\left[\frac{\pi}{6},\ \operatorname{arctg} 2\right]$

(*)

↪ $\sin\frac{\pi}{6}$

$$\Rightarrow \int_M f(x,y)\, d\lambda^2(x,y) \overset{3.15\ (\text{polární souř.})}{=} \int_{\{(r,\alpha)\in G;\ r\in[\cos\alpha,\, 3\cos\alpha],\ \alpha\in[\frac{\pi}{6},\, \operatorname{arctg} 2]\}} \frac{1}{r^4}\cdot r \; d\lambda^2(\alpha, r)$$



(*) • $r\cos\alpha \le r^2 \le 3r\cos\alpha$ ... $r\in[\cos\alpha,\ 3\cos\alpha]$

$$\overset{\text{FUBINI}}{=} \int_{\frac{\pi}{6}}^{\operatorname{arctg}2} \int_{\cos\alpha}^{3\cos\alpha} \frac{1}{r^3}\, dr\, d\alpha = \int_{\frac{\pi}{6}}^{\operatorname{arctg}2} \left[-\frac{1}{2r^2}\right]_{\cos\alpha}^{3\cos\alpha} d\alpha$$

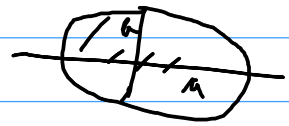
$$= -\frac{1}{2}\int_{\frac{\pi}{6}}^{\operatorname{arctg}2} \frac{1}{9\cos^2\alpha} - \frac{1}{\cos^2\alpha}\, d\alpha = \frac{4}{9}\int_{\frac{\pi}{6}}^{\operatorname{arctg}2} \frac{1}{\cos^2\alpha}\, d\alpha$$

$\frac{1}{9} - 1 = -\frac{8}{9}$

$$= \frac{4}{9}\left[\operatorname{tg}\alpha\right]_{\frac{\pi}{6}}^{\operatorname{arctg}2} = \frac{4}{9}\left(2 - \frac{1}{\sqrt{3}}\right).$$

Př.

$$M = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$



Spočtěte $\lambda^2(M)$:

• M je měřitelná (je uz. $\Rightarrow$ borel.)

$$\Rightarrow \lambda^2(M) = \int_M 1\, d\lambda^2 = \left| \begin{array}{l} \text{zob. pol. souř.} \to 3.15 \\ x = a\, r\cos\alpha \\ y = b\, r \sin\alpha \end{array} \right|$$

$$= \int_{\{(r,\alpha)\in G;\ r^2\le 1\}} ab\cdot r\cdot 1\, d\lambda^2(r,\alpha) \overset{\text{FUBINI}}{=} \int_{-\pi}^{\pi}\int_0^1 ab\, r\, dr\, d\alpha$$

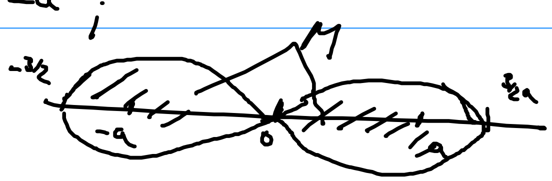
$$= ab\int_{-\pi}^{\pi} \frac{1}{2}\, d\alpha = \frac{ab}{2}\cdot 2\pi = ab\pi.$$

Př.

$$M = \left\{ (x^2+y^2)^2 \le 2a^2(x^2-y^2) \right\} \qquad (a>0)$$

Tzv.: F_1, F_2 ... 2 body ve vzdál. $2a$;

$P\in M \iff PF_1 \cdot PF_2 = a^2$



• M je uz. $\Rightarrow$ borel. $\Rightarrow$ měřitelná!

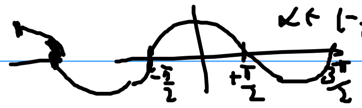
$\rightarrow$ POLÁRNÍ SOUŘADNICE (3.15)

$$\lambda^2(M) = \int_M 1\, d\lambda^2(x,y) = \int_{\{(r,\alpha)\in G;\ r^4 \le 2a^2 r^2(\underbrace{\cos^2\alpha - \sin^2\alpha}_{\cos(2\alpha)})\}} 1\cdot r\, d\lambda^2(r,\alpha)$$

$\Gamma.\ \{(r,\alpha)\in G;\ r^2 \le 2a^2\cos(2\alpha)\}$

$\cos(2\alpha) > 0 \iff 2\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \cup \left(-2\pi, -\frac{3\pi}{2}\right)$

$\alpha \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ $\qquad \alpha \in \left(\frac{3\pi}{4}, \pi\right)$ $\qquad \alpha \in \left(-\pi, -\frac{3\pi}{4}\right)$



$\iff \alpha \in \left(-\pi, -\frac{3\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$

$= \{(r,\alpha)\in G;\ \alpha \in \ldots,\ r \in [0, a\sqrt{2}\sqrt{\cos(2\alpha)})\}$

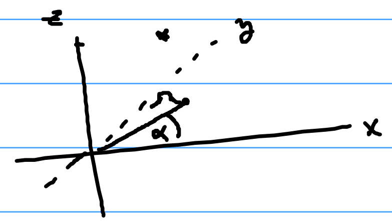
$$\overset{\text{FUBINI}}{=} \int_{\left(-\pi, -\frac{3\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)} \int_0^{a\sqrt{2}\sqrt{\cos(2\alpha)}} r\, dr\, d\alpha$$

$$= \int_{\ldots} a^2 \cos(2\alpha)\, d\alpha = \frac{a^2}{2}\left([\sin(2\alpha)]_{-\pi}^{-\frac{3\pi}{4}} + [\sin(2\alpha)]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + [\sin(2\alpha)]_{\frac{3\pi}{4}}^{\pi} \right)$$

$= \ldots = \underline{\underline{2a^2}}.$

SUBSTITUCE VE 3D

VÁLCOVÉ SOUŘADNICE:



DK VĚTY 3.16: $\boxed{\text{DK PŘÍPADU } a=b=1}$, tj. $\varphi(r,\alpha,z) = (r\cos\alpha,\ r\sin\alpha,\ z)$

• φ je PROSTÉ ... plyne snadno z 3.15

[PROTOŽE V 3.15: $\varphi = (\varphi_1, \varphi_2)$
A ZDE MÁME: $\varphi = (\varphi_1, \varphi_2, z)$]

• $\varphi \in \mathcal{C}^1(G)$ zřejmé, z důkazu 3.15 máme

$$\varphi(G) \supseteq \mathbb{R}^3 \setminus \big(\underbrace{(-\infty,0]\times\{0\}\times\mathbb{R}}_{\text{MÁ } \lambda^3\text{-MÍRU NULA}}\big)$$

tedy $\lambda^3(\varphi(G)^c) = 0$

• $$|J_\varphi|_{\det} = \begin{vmatrix} \cos\alpha & -r\sin\alpha & 0 \\ \sin\alpha & r\cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\cdot\begin{vmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{vmatrix} = r\ (\neq 0)$$

$$\overset{3.14}{\Longrightarrow} \int_{E\cap\varphi(G)} f(x,y,z)\, d\lambda^3(x,y,z) = \int_{\varphi^{-1}(E)} f(r\cos\alpha,\ r\sin\alpha,\ z)\cdot r\, d\lambda^3(r,\alpha,z)$$

$\varphi^{-1}(E) = \varphi^{-1}(E\cap\varphi(G))$

(II)

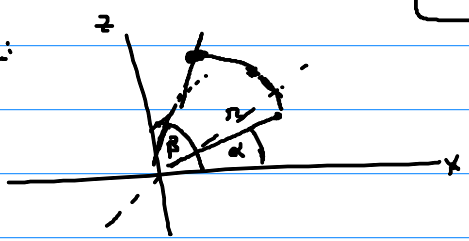
$\lambda^3(E\setminus\varphi(G)) = 0$

$$\int_E f(x,y,z)\, d\lambda^3(x,y,z)$$

■

$\boxed{\text{pozn: } x^2+y^2 = r^2}$

SFÉRICKÉ SOUŘADNICE:



r, α, β

$$\varphi(r, \alpha, \beta) = (r \cos\alpha \cos\beta,\ r \sin\alpha \cos\beta,\ r \sin\beta)$$

POZNÁMKY K DK VĚTY 3.17:

! $G = \{ r > 0;\ \alpha \in (-\pi, \pi);\ \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \}$

! $\varphi(G) \supseteq \mathbb{R}^3 \setminus ((-\infty, 0] \times \{0\} \times \mathbb{R})$

! $\boxed{x^2+y^2+z^2} = r^2 (\underbrace{\cos^2\alpha \cos^2\beta + \sin^2\alpha \cos^2\beta}_{= \cos^2\beta} + \sin^2\beta) = \boxed{r^2}$

PŘÍKLADY

$$M = \{ \sqrt{x^2+y^2+z^2} \le R \} \qquad (R > 0)$$

- M je uz. $\Rightarrow$ borel. $\Rightarrow$ měř.
- $$\lambda^3(M) = \int_M 1 \, d\lambda^3 \underset{\substack{\text{SUBSTITUCE} \\ \text{(SFÉRICKÉ SOUŘ.)}}}{=} \int_{\{(r,\alpha,\beta);\ 0 < r \le R,\ \alpha \in (-\pi,\pi),\ \beta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}} 1 \cdot r^2 \cos\beta \, d\lambda^3(r,\alpha,\beta)$$

$$\overset{\text{FUBINI}}{=} \int_0^R \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\pi/2} r^2 \cos\beta \, d\beta \, d\alpha \, dr$$

$$= \int_0^R r^2 \int_{-\pi}^{\pi} \underbrace{[\sin\beta]_{-\frac{\pi}{2}}^{\pi/2}}_{=2} d\alpha \, dr = 4\pi \int_0^R r^2 dr$$

$$= \frac{4}{3} \pi R^3.$$

$$M = \{ x^2+y^2+z^2 \le 2az,\ x^2+y^2 \le z^2 \}$$

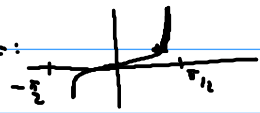
- M je uz. $\Rightarrow$ borel. $\Rightarrow$ měř.
- $$\lambda^3(M) = \int_M 1 \, d\lambda^3(x,y,z) \underset{\substack{\text{SUBSTITUCE} \\ \text{(SFÉR. SOUŘ.)}}}{=} \int_{\{(r,\alpha,\beta) \in G;\ r^2 \le 2a r \sin\beta,\ r^2\cos^2\beta \le r^2 \sin^2\beta\}} 1 \cdot r^2 \cos\beta \, d\lambda^3(r,\alpha,\beta)$$

Γ $\{(r,\alpha,\beta)\in G;\ r \le 2a \sin\beta,\ 1 \le \operatorname{tg}^2 \beta\} =$

$$= \{(r,\alpha,\beta);\ r \le 2a\sin\beta,\ \beta \in (\overset{=\pi/4}{\operatorname{arctg} 1}, \pi/2)\}$$

$[0 < r \le 2a \sin\beta \Rightarrow \sin\beta > 0 \Rightarrow \beta \in (0, \pi/2)]$;

$[|\operatorname{tg}\beta| \ge 1$ obr.:



$]$

$$\overset{\text{FUBINI}}{=} \int_{-\pi}^{\pi} \int_{\frac{\pi}{4}}^{\pi/2} \int_0^{2a\sin\beta} r^2 \cos\beta \, dr \, d\beta \, d\alpha$$

$$= \int_{-\pi}^{\pi} \int_{\frac{\pi}{4}}^{\pi/2} \cos\beta \, \frac{8a^3 \sin^3\beta}{3} \, d\beta \, d\alpha$$

$$= \frac{8a^3}{3} \int_{-\pi}^{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\beta \sin^3\beta \, d\beta \, d\alpha$$

$$= \left| \begin{matrix} t = \sin\beta \\ dt = \cos\beta \, d\beta \end{matrix} \right| = \frac{8a^3}{3} \int_{-\pi}^{\pi} \int_{\frac{1}{\sqrt{2}}}^{1} t^3 \, dt \, d\alpha$$

$$= \frac{8a^3}{3} \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{4} - \frac{1}{16}\right)}_{= \frac{3}{16}} d\alpha = \frac{8a^3}{3} \cdot \frac{3}{16} \cdot 2\pi = \underline{\underline{a^3 \pi}}.$$

Γ

$$M = \{x^2 + y^2 + z^2 \le R^2;\ \underbrace{x^2 + y^2 \le Rx}_{(x - \frac{R}{2})^2 + y^2 \le \frac{R^2}{4}}\}$$

obr.:

- M uz. $\Rightarrow$ borel $\Rightarrow$ měř.
- $\lambda^3(M) = \int_M 1 \, d\lambda^3 \overset{\text{válcové souř.}}{=} \int_{\{(r,\alpha,z)\in G;\ r^2 + z^2 \le R^2,\ r^2 \le R r \cos\alpha\}} 1 \cdot r \, d\lambda^3(r,\alpha,z)$

Γ určení mezí:

- $0 < r \le R\cos\alpha \Rightarrow \cos\alpha > 0 \Rightarrow \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- $z^2 \le R^2 - r^2 \Rightarrow -\sqrt{R^2 - r^2} \le z \le \sqrt{R^2 - r^2}$ $]$

FUBINI
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{R\cos\alpha} \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r \, dz \, dr \, d\alpha$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{R\cos\alpha} r \, 2\sqrt{R^2-r^2} \, dr \, d\alpha$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{R\cos\alpha} r\sqrt{R^2-r^2} \, dr \, d\alpha = \left| \begin{matrix} t = R^2 - r^2 \\ dt = -2r \, dr \end{matrix} \right|$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{R^2(\underbrace{1-\cos^2\alpha}_{\sin^2\alpha})}^{R^2} \sqrt{t} \, dt \, d\alpha = \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(R^3 - R^3|\sin^3\alpha|\right) d\alpha$$

$$= \frac{2}{3}R^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - |\sin^3\alpha|\right) d\alpha \underset{\substack{\downarrow \\ |\sin\alpha| \text{ je sudá fce}}}{=} \frac{2}{3}R^3\left(\pi - 2\int_0^{\pi/2} \sin^3\alpha \, d\alpha\right)$$

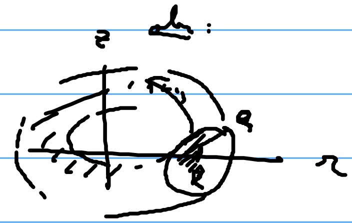
$$= \frac{2}{3}R^3\left(\pi - 2 \cdot \int_0^{\pi/2} \sin\alpha\,(1-\cos^2\alpha)\, d\alpha\right)$$

$$= \cancel{\frac{2}{3}R^3} \left| \begin{matrix} u = \cos\alpha \\ du = -\sin\alpha \, d\alpha \end{matrix} \right| = \frac{2}{3}R^3\left(\pi - 2\underbrace{\int_0^1 (1-u^2)\, du}_{1-\frac{1}{3} = \frac{2}{3}}\right)$$

$$= \underline{\underline{\frac{2}{3}R^3\left(\pi - \frac{4}{3}\right)}}.$$

Př.

$$M = \left\{ \left(R - \sqrt{x^2+y^2}\right)^2 + z^2 \le a^2 \right\}$$



- M je uz. $\Rightarrow$ kompl. $\Rightarrow$ měř.
- $$\lambda^3(M) = \int_M 1 \, d\lambda^3 \underset{\substack{\downarrow \\ \text{válcové} \\ \text{souřadnice}}}{=} \int_{\{(r,\alpha,z)\in G;\ (R-r)^2 + z^2 \le a^2\}} 1 \cdot r \, d\lambda^3(r,\alpha,z)$$

Výpočet mezí:

Pokud $\underbrace{|z| \le a}$, pak $(R-r)^2 \le a^2 - z^2$ lze pro $|z| \le a$

$$|R - r| \le \sqrt{a^2 - z^2}$$

$$-\sqrt{\cdots} \le R - r \le \sqrt{\cdots}$$

$$R - \sqrt{\cdots} \le r \le R + \sqrt{\cdots} \quad \rfloor$$

$$\overset{\text{FUBINI}}{=} \int_{-\pi}^{\pi} \int_{-a}^{a} \int_{R-\sqrt{a^2-z^2}}^{R+\sqrt{\cdots}} r \, dr \, dz \, d\alpha$$

$$= \int_{-\pi}^{\pi} \int_{-a}^{a} \frac{1}{2} \Big(\underbrace{(R+\sqrt{\cdots})^2 - (R-\sqrt{\cdots})^2}_{= 4R\sqrt{\cdots}} \Big) \, dz \, d\alpha$$

$$\doteq 2R \int_{-\pi}^{\pi} \int_{-a}^{a} \sqrt{a^2 - z^2} \, dz \, d\alpha = 2aR \int_{-\pi}^{\pi}\int_{-a}^{a} \sqrt{1-\left(\frac{z}{a}\right)^2} \, dz \, d\alpha$$

$$= \left| \begin{array}{l} t = \frac{z}{a} \\ dt = \frac{dz}{a} \end{array} \right| = 2a^2 R \int_{-\pi}^{\pi} \int_{-1}^{1} \sqrt{1-t^2} \, dt \, d\alpha$$

$$\underset{\sqrt{1-t^2} \text{ JE SUDÁ}}{=} 2a^2 R \cdot 2 \cdot \int_{-\pi}^{\pi} \int_0^1 \sqrt{1-t^2} \, dt \, d\alpha$$

$$= \left| \begin{array}{l} t = \sin u \\ dt = \cos u \, du \end{array} \right| = 4a^2 R \int_{-\pi}^{\pi} \int_0^{\pi/2} \underbrace{\sqrt{1-\sin^2 u}}_{\cos^2 u} \cos u \, du \, d\alpha$$

$$= 4a^2 R \int_{-\pi}^{\pi} \int_0^{\pi/2} \cos^2 u \, du \, d\alpha$$

$$= 4a^2 R \int_{-\pi}^{\pi} \left(\frac{\pi}{4} + \frac{1}{2} \underbrace{\left[\frac{\sin(2u)}{2} \right]_0^{\pi/2}} \right) d\alpha$$

$\ulcorner \cos(2x) = c^2(x) - s^2(x) = 2c^2(x) - 1$

$\Rightarrow c^2(x) = \dfrac{1+\cos(2x)}{2} \quad \lrcorner$

$$= \cancel{4}a^2 R \int_{-\pi}^{\pi} \frac{\pi}{\cancel{4}} \, d\alpha$$

$$= \underline{\underline{2\pi^2 a^2 R}}.$$

POZN:

KDYBYCHOM TO UDĚLALI OPAČNĚ, TJ.

$|R - r| \le a \quad \dots \quad z \in [-\sqrt{a^2 - (R-r)^2}, \sqrt{\cdots}\,]$,

VYŠEL BY SLOŽITĚJŠÍ INTEGRÁL:

$$\int_{\cdots} 2r \sqrt{a^2 - (R-r)^2} \, dr$$

DOKONČENÍ VÍCEROZMĚRNÉ INTEGRACE A LS-INTEGRÁL

Př.

$$M = \left\{ (x,y,z) \in \mathbb{R}^3 ;\ \frac{y^2}{2} + \frac{z^2}{3} \le \frac{1}{\underbrace{x^2+2x+3}_{(x+1)^2+2}} \right\}$$

- M je mz. $\Rightarrow$ borel. $\Rightarrow$ měř.
- $\lambda^3(M) = \int_M 1 \, d\lambda^3 = (*)$

ZOB. VÁLC. SOUŘ.:

$$y = \sqrt{2}\, r \cos\alpha$$
$$z = \sqrt{3}\, r \sin\alpha$$
$$x = w - 1$$

$$\varphi(r, \alpha, w) = (w-1, \sqrt{2}\, r\cos\alpha, \sqrt{3}\, r \sin\alpha)$$

$$(*) = \int_{\{(r,\alpha,w) \in G;\ r^2 \le \frac{1}{w^2+2}\}} \sqrt{6}\, r \, d\lambda^3(r,\alpha,w)$$

$$\overset{\text{FUBINI}}{=} \sqrt{6} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_0^{\sqrt{\frac{1}{w^2+2}}} r \, dr \, dw \, d\alpha$$

$$= \frac{\sqrt{6}}{2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{1}{w^2+2}}_{= \frac{1}{2\left(\left(\frac{w}{\sqrt{2}}\right)^2+1\right)}} dw \, d\alpha$$

$$= \left| \begin{array}{l} t = \frac{w}{\sqrt{2}} \\ dt = \frac{dw}{\sqrt{2}} \end{array} \right| = \frac{\sqrt{6}}{2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \frac{1}{2} \sqrt{2} \frac{1}{t^2+1} \, dt \, d\alpha$$

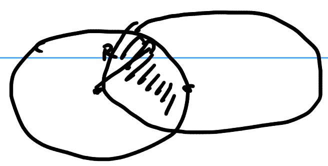
$$= \frac{\sqrt{12}}{4} \int_{-\pi}^{\pi} \underbrace{\left[\operatorname{arctg} t\right]_{-\infty}^{\infty}}_{= \frac{\pi}{2} + \frac{\pi}{2} = \pi} d\alpha$$

$$= \frac{2\sqrt{3}}{4} \pi \cdot 2\pi = \underline{\underline{\sqrt{3} \cdot \pi^2}}.$$

Př.

$$\int_M z^2 \, d\lambda^3, \quad M = \left\{ x^2+y^2+z^2 \le R^2,\ \underbrace{x^2+y^2+z^2 \le 2Rz}_{x^2+y^2+(z-R)^2 \le R^2} \right\}$$

d.:



1. ZP:

$\boxed{\text{SFÉRICKÉ SOUŘ.}}$

$$= \int_{G\cap\{r^2\le R^2,\ r^2 \le 2Rr\sin\beta\}} (r\sin\beta)^2 \cdot r^2 \cos\beta \; d\lambda^3 =$$

$\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\sin\beta \geq \frac{r}{2R} \Rightarrow \beta \geq \arcsin(\dots)$

FUBINI R

$$= \int_0^R \int_{\pi/2}^{\arcsin\left(\frac{r}{2R}\right)} \int_{-\pi}^{\pi} r^4 \sin^2\beta\cos\beta \; d\alpha\, d\beta\, dr$$

$$= 2\pi \int_0^R r^4 \int_{\arcsin\left(\frac{r}{2R}\right)}^{\pi/2} \sin^2\beta \cos\beta \; d\beta\, dr$$

$$= \left| \begin{array}{l} t=\sin\beta \\ dt=\cos\beta\, d\beta \end{array} \right| = 2\pi \int_0^R r^4 \int_{\frac{r}{2R}}^{1} t^2\, dt\; dr$$

$$= \frac{2\pi}{3} \int_0^R r^2 \left(1-\left(\frac{r}{2R}\right)^3\right) dr = \dots = \underline{\underline{\pi R^5 \frac{59}{480}}}.$$

2. ZP.:

$\underbrace{\text{VÁLCOVÉ SOUŘ.}}$

$$= \int_{\{r^2+z^2\le R^2;\ r^2+(z-R)^2\le R^2\}} z^2 \cdot r \; d\lambda^3 = (*)$$

$\boxed{\text{ZPŮSOB 2a}}$:

ZAFIXUJ r ... • $z^2 \le R^2 - r^2$, tj. $\underline{r\le R}$

& $|z| \le \underline{\sqrt{R^2-r^2}}$

• $(z-R)^2 \le R^2 - r^2$, tj. $|z-R| \le \sqrt{R^2-r^2}$

$|z-R| = R-z$, tj.

$z \geq \underline{R-\sqrt{R^2-r^2}}$

Celkem: $z \in \left[R-\sqrt{R^2-r^2},\ \sqrt{R^2-r^2}\right] =: I$

ale $I \neq \emptyset \iff R \le 2\sqrt{R^2-r^2}$

$\iff \dots \iff r \le \frac{3}{4}R$

$$\Rightarrow (*) = 2\pi \int_0^{3/4 R} \int_{R-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r z^2 \; d\lambda^3 =$$

= ... dost složitý integrál ...

Způsob 2B

Zafixuji z .... pak • $r^2 \le R^2 - z^2 \implies z \le R$

• $r^2 \le 2Rz - z^2 \implies z \le 2R$

Zároveň $2Rz - z^2 \le R^2 - z^2 \iff \dots$

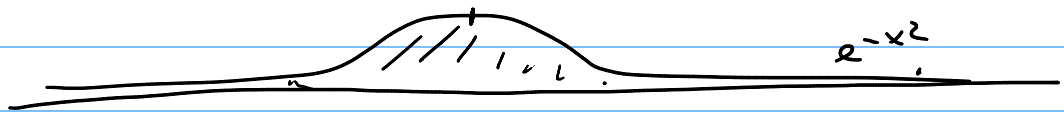
$\dots \iff z \le \frac{R}{2}$

$$\implies (*) = 2\pi \int_0^{R/2} \int_0^{\sqrt{2Rz - z^2}} r z^2 \, dr \, dz$$

$$+ 2\pi \int_{R/2}^{R} \int_0^{\sqrt{R^2 - z^2}} r z^2 \, dr \, dz$$

$$= \dots = \underline{\frac{59}{480} \pi R^5}.$$

d.:



Př

$$\int_0^\infty e^{-x^2} \, dx =: I$$

$\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} \, dx \, dy$

Trik:

$$\bullet \int_{(0,\infty)^2} e^{-x^2} e^{-y^2} \, d\lambda^2(x,y) \overset{\text{Fubini}}{=} \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)$$

$$= I^2$$

Na druhou stranu,

$x, y > 0$, tj. $r \cos\alpha > 0$, $r \sin\alpha > 0$

$$= [\text{polární souřadnice}] = \int_{\{r \in (0,\infty),\, \alpha \in (0, \frac{\pi}{2})\}} e^{-r^2} \cdot r \, d\lambda^2(r,\alpha)$$

$$\overset{\text{Fubini}}{=} \int_0^{\frac{\pi}{2}} \int_0^\infty r e^{-r^2} \, dr \, d\alpha = \int_0^{\pi/2} \left[e^{-r^2} \left(-\frac{1}{2}\right) \right]_0^\infty d\alpha$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{2}\right)(0 - 1) = \frac{\pi}{4}$$

$$\implies I^2 = \frac{\pi}{4}$$

LS- INTEGRÁL

MOTIVACE: • $\boxed{\text{ZATÍM}}$: „(L)-$\int$" ≈ POČÍTÁNÍ OBJEMU /OBSAHU

• JINÉ MÍRY SE PŘIROZENĚ VYSKYTUJÍ V PRAXI

[„$P(x \in A)$ ≈ $\frac{\mu(A)}{\mu(X)}$", kde $A \in \mathcal{A}$ a $(X,\mathcal{A})$ měř. prostor

„$P(f(x) \in A)$ ≈ $\boxed{\int_X f(x)\, d\mu(x)}$" $\frac{}{\mu(X)}$ → $\int_X f \, d\mu$]

LSFI – PŘÍKLADY: $A, B \in \mathbb{R}$

a) $$m(I) := \frac{\ell(I \cap [A,B])}{\ell([A,B])}$$ pro $I \subseteq \mathbb{R}$ interval

[m ≈ pravděpodobnost že náhodné číslo z $[A,B]$ je v intervalu I]

b) $$m(I) = \begin{cases} 0 \dots & \{1,-1\} \cap I = \emptyset \\ 1/2 \dots & |I \cap \{1,-1\}| = 1 \\ 1 \dots & \{1,-1\} \subseteq I \end{cases}$$ pro $I \subseteq \mathbb{R}$ interval

MOHUTNOST

[m ≈ pstb že náhodně zvolené číslo z $\{1,-1\}$ bude v intervalu I

~~[illegible]~~ SIMULUJE HOD MINCÍ ...]

3.18: $m: \mathcal{I} \to [0,\infty)$ je LSFI $\iff \exists\, \varphi$ rostoucí spojitá zprava:

tž $m((a,b]) = \varphi(b) - \varphi(a)$

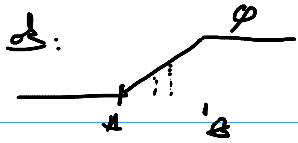
NÁČRTEK DK.: „$\Rightarrow$" $\varphi(x) := \begin{cases} m((0, x]) & \text{pro } x \geq 0 \\ -m((x,0]) & \text{pro } x < 0 \end{cases}$

a ověří se ...

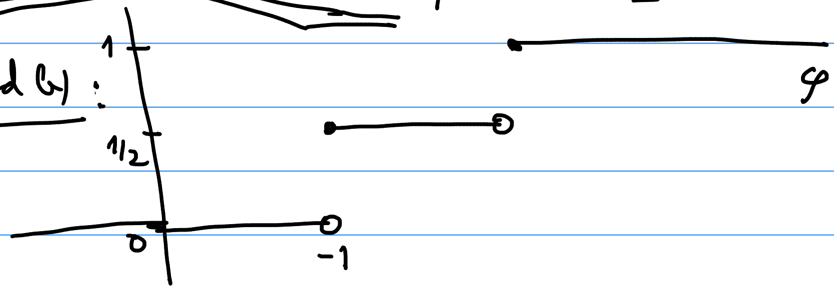
„$\Leftarrow$" těžší ověření ... $\boxtimes$

Př.: ad a) ... $\varphi(x) = \begin{cases} 0 & \dots\ x < A \\ \frac{x-A}{B-A} & \dots\ x \in [A,B] \\ 1 & \dots\ x > B \end{cases}$

ad:



ad b):



Pozn.:

- LSFI definujeme na intervalech typu $(a,b]$ značí tvrzení 3.18 $\lceil F(b-) - F(a+) \neq F(b) - F(a+)$ atd.. $\rfloor$
- $(\, , \,]$ výhodné, protože systém těchto int. uz. na $\cap, \cup, \setminus$

  $\lceil$ pokud $(a,b] \cap (c,d]$ je interval, tak je stejného typu $\rfloor$

  ... šlo by i „$[\, , \,)$" ... to je otázka výběru
- zprava spojitost: kvůli tomu, že pro $b_j \downarrow b$

$$\bigcap_{b_j} (a, b_j] = (a,b]$$

tj. chceme, aby $\lim_j m(a,b_j] = m((a,b])$

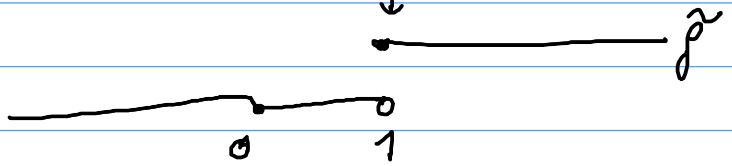
tj. „zprava spojitost"

VÝPOČET LS-INTEGRÁLU

od POZN na str. 14: $\quad f(x) = \begin{cases} 1 & \dots\ x = 1 \\ 0 & \dots\ x \in [0,1) \end{cases}$

Pak

$$(LS)-\int_0^1 f(x)\,df(x) = (LS)-\int_{[0,1]} f(x)\,d\tilde f(x) = \int_{[0,1)} 0\,d\tilde f(x) + \int_{\{1\}} f(x)\,d\tilde f(x)$$



$$= 0 + \mu_{\tilde f}(\{1\})\, f(1) = \lim_{n\to\infty}\left(\tilde f(1) - \tilde f\left(1-\tfrac{1}{n}\right)\right) f(1)$$

$$= 1 \cdot 1 = \underline{1}.$$

ale $(RS)-\int_0^1 f(x)\,df(x)$ neex. protože $\underline{S}_\varphi(0,1,D) = 0$ pro každé dělení D

ale $\overline{S}_\varphi(0,1,D) = 1$ pro dělení $D = \{0,1\}$

DŮKAZ FAKTU 3.23: $\varphi : \mathbb{R} \to \mathbb{R}$ neklesající, $\tilde\varphi(x) := \varphi(x+)$; $a < b \in \mathbb{R}$

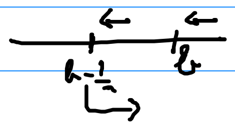
(i) $\mu_\varphi((a,b]) = \varphi(b+) - \varphi(a+)$ z definice ✓

(ii) $\mu_\varphi(\{b\}) = \mu_\varphi\left(\bigcap_{n=1}^{\infty}\left(b-\tfrac{1}{n}, b\right]\right)$

$$\underset{3.3(iv)}{=} \lim_{n\to\infty} \mu_\varphi\left(\left(b-\tfrac{1}{n}, b\right]\right)$$

$$\overset{(i)}{=} \lim_{n\to\infty} \varphi(b+) - \varphi\left(\left(b-\tfrac{1}{n}\right)+\right) = \varphi(b+) - \varphi(b-)$$

viz. obr. (podrobněji výše (4...)



(iii) $\mu_\varphi([a,b]) = \mu_\varphi(\{a\}) + \mu_\varphi((a,b]) \overset{(i)+(ii)}{=} \cancel{\varphi(a+)} - \varphi(a-) + \varphi(b+) - \cancel{\varphi(a+)}$

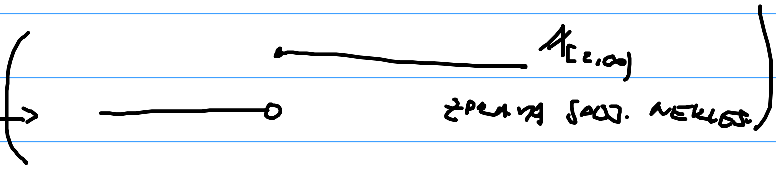
(iv) $\mu_\varphi([a,b)) = \mu_\varphi([a,b]) - \mu_\varphi(\{b\}) \overset{(iii)+(ii)}{=} \cancel{\varphi(b+)} - \varphi(a-) - \cancel{\varphi(b+)} + \varphi(b-)$

(A) $\mu_\varphi((a,b)) = \mu_\varphi((a,b]) - \mu_\varphi(\{b\}) \overset{(i)+(ii)}{=} \cancel{\varphi(b+)} - \varphi(a+) \cancel{-\varphi(b+)} + \varphi(b-))$

☒

PŘÍKLADY

a) $\displaystyle\int_{[2,3]} x^2 \, d\chi_{[2,\infty)}(x)$



1 na $[2,3]$ je spoj. v 3

$$= \int_{\{2\}} x^2 \chi_{[2,\infty)} + \int_2^3 x^2 \, d\overbrace{\chi_{[2,\infty)}}$$

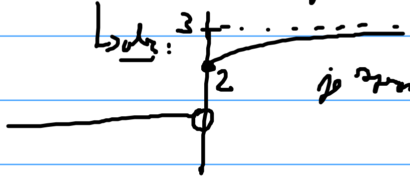
$$= 4\cdot(1-0) + \int_{(2,3)} x^2 \, d\chi_{[2,\infty)} = 4 + 0 = \underline{4}.$$

3.24.5
+3.21.3

3.24.4

b) $\displaystyle\int_{[0,\infty)} e^x \, d\Big(\overbrace{(3-e^{-2x})\chi_{[0,\infty)}}^{=:f(x)}\Big) =$

Graf:



je zprava spoj. neklesající.

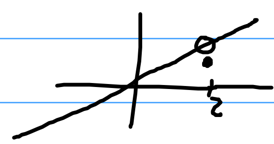
$$= \int_{\{0\}} e^x \, df(x) + \int_{(0,\infty)} e^x \, df(x) =$$

$$= e^0\cdot\big(f(0+)-f(0-)\big) + \int_0^\infty e^x \cdot \Big(\underbrace{(3-e^{-2x})'}_{=2e^{-2x}}\Big) \, dx$$

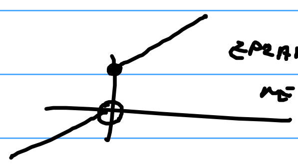
$$= 1\cdot(2-0) + 2\int_0^\infty e^{-x} \, dx = 2 + 2\left[-e^{-x}\right]_0^\infty$$

$$= 2 + 2\cdot(-0+1) = \underline{\underline{4}}.$$

c) $\displaystyle\int_{[1,3]} f(x) \, dg(x)$, kde $f(x) = \begin{cases} x & \dots x \neq 2 \\ 1 & \dots x = 2 \end{cases}$ Graf:



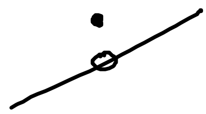
$g(x) = \begin{cases} x & \dots x < 0 \\ x+1 & \dots x \geq 0 \end{cases}$



$$\underset{\substack{g \text{ spoj. v } 1 \\ a\ v\ 3}}{=} \int_{(1,3)} f(x) \, d\overbrace{g(x)}^{=x+1} = \int_1^3 f(x)\cdot 1 \, dx \underset{f(x)=x \text{ s.v.}}{=} \int_1^3 x \, dx = \left[\frac{x^2}{2}\right]_1^3$$

$$= \underline{\underline{4}}.$$

d) $\displaystyle\int_{[-1,1]} f(x)\, dg(x)$, kde $f(x) = \begin{cases} x & \dots\ x \neq 0 \\ 1 & \dots\ x = 0 \end{cases}$



$g(x)$ jako v c)

g skok v ± 1

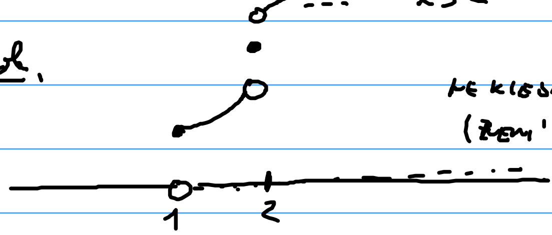
$$= \int_{(-1,0)} f(x)\, d\overbrace{g(x)}^{=x} + \int_{\{0\}} f(x)\, dg(x) + \int_{(0,1)} f(x)\, d\overbrace{g(x)}^{=x+1}$$

$$= \int_{-1}^{0} \overbrace{f(x)}^{x} dx + f(0)\big(g(0+)-g(0-)\big) + \int_0^1 \overbrace{f(x)}^{x} dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{0} + 1.(1-0) + \left[\frac{x^2}{2}\right]_0^1 = \dots = \underline{\underline{1}}.$$

e) $\displaystyle\int_{[0,3)} x^2\, d\varphi(x)$, $\varphi(x) = \begin{cases} 0 & \dots\ x < 1 \\ x^2-2x+2 & \dots\ x \in [1,2) \\ 3 & \dots\ x = 2 \\ x+2 & \dots\ x > 2 \end{cases}$

$\underline{\mathscr{d}}.$



neklesající (není zrovna skok.)

$$= \underbrace{\int_{[0,1)} x^2\, d0}_{=0} + \int_{\{1\}} x^2\, d\varphi(x) + \int_{(1,2)} x^2\, d(x^2-2x+2) + \int_{\{2\}} x^2\, d\varphi(x)$$

$$+ \int_{(2,3)} x^2\, d(x+2)$$

$$= 0 + 1.\underbrace{(\varphi(1+)-\varphi(1-))}_{=1} + \int_1^2 x^2(2x-2)\, dx + 4.\underbrace{(\varphi(2+)-\varphi(2-))}_{=2}$$

$$+ \int_2^3 x^2\, dx = 0 + 1 + 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 + 8 + \left[\frac{x^3}{3}\right]_2^3$$

$$= \dots = \underline{\underline{\frac{109}{6}}}.$$

f) $g(x) = \begin{cases} e^{3x} & \dots\ x \leq 0 \\ 2 & \dots\ x \in (0,1) \\ 2x+1 & \dots\ x \geq 1 \end{cases}$; $f(x) = \begin{cases} e^{-2x} & \dots\ x \leq 1 \\ x & \dots\ x > 1 \end{cases}$

(i) $$\int_{(-1,0)} f(x)\, dg(x) = \int_{(-1,0)} e^{-2x}\, d(e^{3x}) = \int_{-1}^{0} e^{-2x}.\ 3.e^{3x}\, dx = 3\int_{-1}^{0} e^{x}\, dx$$

$$= \underline{\underline{3(1-e^{-1})}}$$

(ii) $\displaystyle\int_{[-1,0]} f(x)\,dg(x) = \int_{(-1,0)} f(x)\,dg(x) + \int_{\{-1\}} f(x)\,dg(x) + \int_{\{0\}} f(x)\,dg(x)$

$$\overset{(i)}{=} 3(1-e^{-1}) + e^{2}(0) + 1\cdot(\overbrace{2-1}^{=1}) = \underline{\underline{4-3e^{-1}}}.$$

(iii) $\displaystyle\int_{(-1,1)} f(x)\,dg(x) \underset{g \text{ cost. } = -1}{=} \int_{[-1,0]} f(x)\,dg(x) + \int_{(0,1)} f(x)\,d\overbrace{g(x)}^{=2} \overset{(ii)}{=} 4-3e^{-1}+0$

$$= \underline{4-3e^{-1}}.$$

(iv) $\displaystyle\int_{(-1,1]} f(x)\,dg(x) = \int_{(-1,1)} f(x)\,dg(x) + \int_{\{1\}} f(x)\,dg(x) \overset{(iii)}{=} 4-3e^{-1} + e^{-2}(3-2)$

$$= \underline{\underline{4-3e^{-1}+e^{-2}}}.$$

(v) $\displaystyle\int_{[1,3]} f(x)\,dg(x) \underset{g \text{ cost. } = 3}{=} \int_{\{1\}} f(x)\,dg(x) + \int_{(1,3)} f(x)\,dg(x) = e^{-2} + \int_1^3 x\cdot 2\,dx$

$$= \underline{\underline{8+e^{-2}}}.$$

(vi) $\displaystyle\int_{(-\infty,0)} f(x)\,dg(x) = \int_{-\infty}^{0} e^{-2x}\,d(e^{3x}) = 3\int_{-\infty}^{0} e^{x}\,dx = 3\left[e^{x}\right]_{-\infty}^{0}$

$$= \underline{3}.$$

KONVERGENCE INTEGRÁLŮ

OPAK – LS INTEGRÁL:

Př.: $\displaystyle\int_{[0,5]} x^2+1 \, d[x]$

$$= \int_{\{0\}} (x^2+1)\, d[x] + \underbrace{\int_{(0,1)} x^2+1 \, d0}_{=0} + \int_{\{1\}} (x^2+1)\, d[x] + \underbrace{\int_{(1,2)} x^2+1\, d0}_{=0}$$

$$+ \int_{\{2\}} x^2+1 \, d[x] + \dots$$

$$= \sum_{i=0}^{5} \int_{\{i\}} x^2+1 \, d[x] = \sum_{i=0}^{5} (i^2+1) = 1 + 2 + 5 + 10 + 17 + 26$$

$$= \underline{\underline{61}}.$$

PŘIPOMEŇ:

- $f \in \mathcal{C}([a,b]) \Rightarrow (L)\text{-}\int_a^b f = (R)\text{-}\int_a^b f = (N)\text{-}\int_a^b f \in \mathbb{R}$
- $f \in \mathcal{C}((a,b)) \Rightarrow \left((L)\text{-}\int_a^b f \iff (L)\text{-}\int_a^b |f| \iff (N)\text{-}\int_a^b |f| \right)$
- $\int_0^1 x^\alpha \, dx$ K. $\iff \alpha > -1$
- $\int_1^\infty x^\alpha \, dx$ K. $\iff \alpha < -1$
- (SK): $\int_a^b g < \infty$, $|f| \leq g \Rightarrow \int_a^b f \in \mathbb{R}$

R.Š.: • $1 \ll \log x \ll x^n \ll a^x$, $x \to \infty$ $(a > 1)$

$\ulcorner f(x) \ll g(x) \overset{\text{def}}{\iff} \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, tj.

$[\exists K > 0 \ \forall x > K: f(x) \leq g(x)] \lrcorner$

- $\log x \ll x^{-\varepsilon}$, $x \to 0^+$

$\ulcorner$ tj. $\lim_{x\to 0^+} x^\varepsilon \log x = 0$, tj. $\exists \delta > 0 \ \forall x \in (0,\delta): |\log x| \leq \frac{1}{x^\varepsilon} \lrcorner$

$\bullet$ (LSK): $\quad f, g \in \mathcal{C}([a,b))$, $\lim_{x \to a^+} \frac{f(x)}{g(x)} \in \mathbb{R} \setminus \{0\}$

$$\text{pa} \quad \int_a^b f \ \text{k.} \iff \int_a^b g \ \text{k.}$$

Př

$$\int_0^\infty \frac{1}{1+x^2} \, dx$$

2 způsoby: 1) $\frac{1}{1+x^2} \in \mathcal{C}([0,\infty))$ ... stačí zjistit konvergenci u ∞

„u ∞": $\quad \frac{1}{1+x^2} \leq \frac{1}{x^2}$, $\quad x \in (0,\infty)$

$$\& \int_1^\infty \frac{1}{x^2} \in \mathbb{R}$$

$$\underset{\text{S.K.}}{\Longrightarrow} \quad \int_1^\infty \frac{1}{1+x^2} \, dx \in \mathbb{R}$$

Celkem: $\quad \int_0^\infty f = \underbrace{\int_0^1 f}_{\in \mathbb{R}} + \underbrace{\int_1^\infty f}_{\in \mathbb{R}}$ je konv.!

2) $\int_0^\infty \frac{1}{1+x^2} \, dx = [\operatorname{arctg} x]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

(b) $\quad \int_0^\infty \frac{1}{e^{x^2}} \, dx$

Máme $\lim_{x \to \infty} \frac{x^2}{e^{x^2}} = 0$, tedy $\exists K>0 \ \forall x>K$:

$$\frac{1}{e^{x^2}} \leq \frac{1}{x^2}$$

S.K. + FAKT ŽE $\int_1^\infty \frac{1}{x^2}$ k. $\Longrightarrow$

$$\int_K^\infty \frac{1}{e^{x^2}} \, dx \quad \text{je konv.}$$

Dále $\frac{1}{e^{x^2}} \in \mathcal{C}([0,K])$, tedy $\int_0^K \frac{1}{e^{x^2}}$ je konv.

Celkem: $\int_0^\infty \frac{1}{e^{x^2}}$ je konv.

(c) $\quad \int_0^1 \frac{1}{\sin x} \, dx$

$\Gamma\ \frac{1}{\sin x} \in \mathcal{C}((0,1])$, tedy stačí vyšetřit jen singularitu „v 0"

„v 0": $\left[\text{IDEA: } \frac{1}{\sin x} \approx \frac{1}{x} \Rightarrow D.\right]$

Máme $\lim_{x\to 0^+} \frac{\frac{1}{\sin x}}{\frac{1}{x}} \overset{\text{Z.L.}}{=} 1 \in \mathbb{R}\setminus\{0\}$

$\overset{\text{LSK+FAKT ŽE}}{\Longrightarrow} \int_0^1 \frac{1}{x}$ D. $\quad \int_0^1 \frac{1}{\sin x}\,dx$ D. $\rfloor$

(d) $\int_{1/2}^{1} \sqrt{\frac{1}{|\lg x|}}\,dx$

$\Gamma\ \sqrt{\frac{1}{|\lg x|}} \in \mathcal{C}([1/2, 1))$ ... stačí vyšetřit jen „v 1"

„v 1": $\left[\text{IDEA: } \sqrt{\frac{1}{\lg x}} \approx \sqrt{\frac{1}{x-1}} \Rightarrow K.\right]$

Máme $\lim_{x\to 1^-} \frac{\sqrt{\frac{1}{|\lg x|}}}{\sqrt{\frac{1}{1-x}}} = \lim_{x\to 1^-} \sqrt{\frac{1-x}{-\lg x}} = \lim_{x\to 1^-} \sqrt{\frac{x-1}{\lg x}}$

$\overset{\text{Z.L.+SPOJ.}}{=} 1 \in \mathbb{R}\setminus\{0\}$

$\overset{\text{LSK + FAKT}}{\Longrightarrow} \int_{1/2}^{1} \frac{1}{\sqrt{1-x}}$ K. $\quad \int_{1/2}^{1} \sqrt{\frac{1}{|\lg x|}}\,dx$ K.

(e) $\int_2^\infty \underbrace{\frac{1}{x^p (\lg x)^q}}_{=:f(x)}\,dx \qquad f \in \mathcal{C}([2,\infty))$ ... stačí vyšetřit „v ∞"

„v ∞":

$p>1$:

- $\frac{1}{x^p(\lg x)^q} \le \frac{1}{(\lg 2)^q}\ \frac{1}{x^p}$, $x \in (2,\infty)$, $q \ge 0$

$\overset{\text{S.K.+FAKT ŽE } \int_2^\infty \frac{1}{x^p}dx \in \mathbb{R}}{\Longrightarrow} \int_2^\infty f(x)\,dx \in \mathbb{R}$

- POKUD $q<0$: FIX $\varepsilon>0$ TAK, ŽE $p-\varepsilon>1$.

Protože $\lim_{x\to\infty} \frac{(\lg x)^{-q}}{x^\varepsilon} = 0$, $\exists k>0$: $(\lg x)^{-q} \le x^\varepsilon$, $x\in(k,\infty)$

tedy $\forall x > K:$ $\quad \frac{1}{x^p (\log x)^q} \leq \frac{1}{x^{p-\varepsilon}}$

$\Longrightarrow$ S.K. + FAKT, ŽE $\int_K^\infty \frac{1}{x^{p-\varepsilon}}\,dx \in \mathbb{R}$ $\qquad \int_K^\infty f(x)\,dx \in \mathbb{R}$

Celkem: $\int_2^\infty f(x)\,dx \in \mathbb{R}$

$p < 1$:

POKUD $q \leq 0$: $\quad \frac{1}{x^p (\log x)^q} \geq \frac{1}{x^p (\log 2)^q}$

$\Longrightarrow$ S.K. + FAKT, ŽE $\int_2^\infty \frac{1}{x^p}$ D. $\qquad \int_2^\infty f(x)\,dx$ D.

POKUD $q > 0$:

FIX $\varepsilon > 0$ ŽE $p + \varepsilon < 1$

pak (analogicky jako výše) $\exists K > 0 \ \forall x > K:$

$(\log x)^q \leq x^\varepsilon$

$\frac{1}{x^p (\log x)^q} \geq \frac{1}{x^{p+\varepsilon}}$

$\Longrightarrow$ S.K. + FAKT ŽE $\int_2^\infty \frac{1}{x^{p+\varepsilon}}\,dx$ D. $\qquad \int_2^\infty f(x)\,dx$ D.

$p = 1$: $\quad \int_2^\infty f(x)\,dx = \left| \begin{matrix} t = \log x \\ dt = \frac{1}{x}\,dx \end{matrix} \right| = \int_{\log 2}^\infty \frac{1}{t^q}\,dt$

je K. $\Longleftrightarrow q > 1$ $\quad$ (tak už víme).

(f) $\int_0^{1/2} \overbrace{\frac{1}{x^p |\log x|^q}}^{=: f(x)}\,dx \quad \left(= \int_0^{1/2} \frac{-1}{x^p (\log x)^q}\,dx \right)$

$f \in \mathcal{C}((0, 1/2])$, tedy stačí zkoumat jen „v 0"

$p < 1$:

POKUD $q \geq 0$:

$$f(x) \leq \frac{1}{x^p |\lg 1/2|^q}, \quad x \in (0, 1/2)$$

$\overset{\text{S.K. + FAKT ŽE } \int_0^{1/2} \frac{1}{x^p} \text{ K.}}{\Longrightarrow}$ $\int_0^{1/2} f(x)\,dx$ K.

POKUD $q \leq 0$:

„$|\lg x|^{-q} << x^{-\varepsilon}$ pro malé $\varepsilon > 0$"

FIX $\varepsilon > 0$ TAK, ŽE $p + \varepsilon < 1$

platí $\lim_{x \to 0^+} x^{\varepsilon} (\lg x)^{-q} = 0$, tedy $\exists \delta > 0$

$\forall x \in (0, \delta)$: $|\lg x|^{-q} \leq x^{-\varepsilon}$, tj.

$\forall x \in (0, \delta)$: $f(x) \leq \frac{x^{-\varepsilon}}{x^p} = \frac{1}{x^{p+\varepsilon}}$

$\overset{\text{S.K. + FAKT, ŽE } \int_0^{1/2} \frac{1}{x^{p+\varepsilon}} dx \in \mathbb{R}}{\Longrightarrow}$ $\int_0^{1/2} f(x)\,dx$ K.

$p > 1$:

FIX $\varepsilon > 0$ TAK, ŽE $p - \varepsilon > 1$

$\boxed{\text{PRO } q \geq 0}$ máme (analogicky jako výše)

$\exists \delta > 0 \ \forall x \in (0, \delta)$: $f(x) \geq \frac{1}{x^p x^{-\varepsilon}} = \frac{1}{x^{p-\varepsilon}}$

$\overset{\text{S.K. + FAKT ŽE } \int_0^{1/2} \frac{1}{x^{p-\varepsilon}} dx \text{ D.}}{\Longrightarrow}$ $\int_0^{1/2} f(x)\,dx$ D.

$\boxed{\text{PRO } q < 0}$

$$\frac{1}{x^p |\lg x|^q} = \frac{|\lg x|^{-q}}{x^p} \geq \frac{|\lg 1/2|^{-q}}{x^p}$$

$\overset{\text{S.K. + FAKT } \int_0^{1/2} \frac{1}{x^p} \text{ D.}}{\Longrightarrow}$ $\int_0^{1/2} f(x)\,dx$ D.

$p = 1$: $\int_0^{1/2} f(x)\,dx = \left| \begin{array}{l} h = \lg x \\ dh = \frac{1}{x} dx \end{array} \right| = \int_{|\lg 1/2|}^{\infty} \frac{1}{h^q} \, dh$

tedy $\int f(x)\,dx$ K. $\Longleftrightarrow q > 1$

g) $\displaystyle\int_0^\infty \underbrace{e^{-x}\, x^{s-1}\, (\log x)^k}_{=: f(x)}\, dx \qquad \left(s \in \mathbb{R},\ k \in \mathbb{N} \cup \{0\} \right)$

$f \in \mathcal{C}((0,\infty))$ ... tedy stačí vyšetřit konv. „v 0" a „v ∞"

„v ∞":

Máme $\displaystyle\lim_{x\to\infty} \frac{x^{s-1}(\log x)^k\, x^2}{e^x} = 0$

$\left(= \displaystyle\lim_{x\to\infty} \left(\frac{\log x}{x}\right)^k \cdot \lim_{x\to\infty} \frac{x^{s-1+2+k}}{e^x} = 0 \cdot 0 = 0 \right)$

tedy $\exists k > 0\ \forall x > k: \quad \dfrac{x^{s-1}(\log x)^k}{e^x} \le \dfrac{1}{x^2}$

$\overset{\text{S.K. + FAKT, ŽE } \int_1^\infty \frac{1}{x^2} \text{ K.}}{\Longrightarrow} \quad \displaystyle\int_1^\infty f(x)\, dx \text{ K.}$

„v 0":

Máme $\displaystyle\lim_{x\to 0^+} \frac{f(x)}{x^{s-1}(\log x)^k} = 1$

$\overset{\text{LSK}}{\Longrightarrow} \left(\displaystyle\int_0^{1/2} f(x) \text{ K.} \iff \int_0^{1/2} x^{s-1}(\log x)^k \text{ K.} \right)$

$\overset{(f)}{\Longrightarrow} \left(\displaystyle\int_0^{1/2} f(x) \text{ K.} \iff s > 0 \text{ nebo } (s = 0 \wedge k < -1) \right)$

$\iff \underline{\underline{s > 0}}$

Celkem: Integrál konv. $\iff s > 0$ (a $k \in \mathbb{N}$)

h) $\displaystyle\int_0^1 \underbrace{x^{p-1}(1-x)^{q-1}}_{=: f(x)}\, dx$

• $f \in \mathcal{C}((0,1))$ ... tedy stačí vyšetřit konv. „v 0" a „v 1"

[IDEA: „v 0": $f \approx x^{p-1}$, tedy K. $\iff p > 0$

„v 1": $f \approx (1-x)^{q-1}$ — " — $\iff q > 0$]

PROHOZENÍ LIMITY A INTEGRÁLU

Př

$$\int_0^1 \underbrace{x^{p-1}\,(1-x)^{q-1}}_{=:f(x)}\,dx$$

$f\in \mathcal{C}((0,1))$ ... stačí vyšetřit kvůli „v 0" a „v 1"

„v 0": Máme $\lim_{x\to 0^+} \frac{f(x)}{x^{p-1}} = 1 \in \mathbb{R}\setminus\{0\}$

$\Rightarrow$ LSK + FAKT, ŽE $\int_0^1 x^{p-1}\,dx \in \mathbb{R} \iff p-1>-1 \iff p>0$

$\underbrace{\text{„v 0" je integrál konečný}}_{\text{tj. } \int_0^{1/2} f(x)\,\in\,\mathbb{R}} \iff p>0.$

Pozn.: $f(x)\geq 0$ VŽDY $\Rightarrow$ INTEGRÁL EX. (VŽDY)

„v 1": Máme $\lim_{x\to 1^-} \frac{f(x)}{(1-x)^{q-1}} = 1 \in \mathbb{R}\setminus\{0\}$

$\Rightarrow$ LSK + FAKT, ŽE $\int_0^1 (1-x)^{q-1} \in \mathbb{R} \iff q-1>-1 \iff q>0$

Tedy, $\underbrace{\text{„v 1" je integrál konečný}}_{\text{tj. } \int_{1/2}^1 f(x)\,\in\,\mathbb{R}} \iff q>0$

Celkem: Integrál konečný $\iff p>0$ & $q>0$

Př

$$\int_0^1 \underbrace{\frac{\log(1-p^2x^2)}{x^2\sqrt{1-x^2}}}_{=:f_p(x)}\,dx \qquad (p\in\mathbb{R})$$

- Aby byl log. def. na $(0,1)$, je třeba $1-p^2x^2>0$ pro $x\in(0,1)$

tedy $p^2 < \frac{1}{x^2}$ pro $x\in(0,1)$

$p^2 < y$ pro $y\in(1,\infty)$

$p^2 < 1$

$\Rightarrow$ $p\in[-1,1]$.

• $p=0$ ... jde integrál tzj (je rovna nule)

• $f_p(x) \in \mathcal{C}((0,1))$ pro každé $p \in [-1,1]$ ... stačí tedy zjistit kon.

„v 0" a „v 1"

„v 0":
$$\lim_{x\to0^+} f(x) \overset{AR}{=} \lim_{x\to0^+} \frac{1}{\sqrt{1-x^2}} \lim_{x\to0^+} \frac{\log(1-p^2x^2)}{-p^2x^2} \frac{-p^2\cancel{x^2}}{\cancel{x^2}}$$

$$\overset{VoLSF+Z.L.}{=} 1\cdot1\cdot(-p^2) \in \mathbb{R}\setminus\{0\} \text{ pro } p\neq0$$

$\Rightarrow$ LSK + FAKT ŽE $\int_0^1 1 \in \mathbb{R}$

$\int_0^{1/2} f(x)\,dx$ je konečné pro $p\in[-1,1]\setminus\{0\}$

„v 1": [IDEA: $f(x) \approx \frac{\log(1-p^2x^2)}{\sqrt{1-x^2}} \approx$ $p\neq\pm1$: $\approx \frac{1}{\sqrt{1-x}}$ ... kon; $|p|=1$: $\approx \frac{\log(1-x)+\log(1+x)}{\sqrt{1-x}}$ ... kon

... DETAILY DCV

Celkem: integrál kon $\iff p\in[-1,1] \iff$ integrál ex.

Př.
$$\int_0^{\pi/2} \underbrace{(\tan x)^p}_{=:f(x)}\,dx \qquad (p\in\mathbb{R})$$

• $f\in\mathcal{C}((0,\frac{\pi}{2}))$, $f\geq0$ ... integrál ex. pro $p\in\mathbb{R}$,

konvergenci stačí zjistit „v 0" a „v $\frac{\pi}{2}$"

• „v 0": [IDEA: Víme: $\lim_{x\to0}\frac{\tan x}{x}=1$, tedy $f(x)\approx x^p$

$\rightsquigarrow$ kon $\iff p>-1$

• „v $\pi/2$": IDEA:
$$\lim_{x\to\frac{\pi}{2}^-}\frac{\tan x}{\left(\frac{\pi}{2}-x\right)^{-1}} \overset{c^4}{=} \cancel{\lim_{x\to\frac{\pi}{2}^-}} \frac{1}{\cancel{\cos x}} \lim_{x\to\frac{\pi}{2}^-} \sin x \cdot \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x\left(\frac{\pi}{2}-x\right)^{-1}}$$

$$= 1\cdot\lim_{x\to\frac{\pi}{2}^-}\frac{\frac{\pi}{2}-x}{\cos x} \overset{c^4\left(\frac{0}{0}\right)}{=} \lim_{x\to\frac{\pi}{2}^-}\frac{-1}{-\sin x} = 1 \in\mathbb{R}\setminus\{0\}$$

$\Rightarrow$ LSK + Fakt že $\int_{1/2}^{\pi/2} \left(\frac{\pi}{2}-x\right)^{-p} dx$ k. $\iff -p > -1 \iff p<1$

$$\int_{1/2}^{\pi/2} f_{\alpha,1} \text{ kg } \iff p<1$$

Celkem: $\int_{0}^{\pi/2} f_{\alpha,1}$ kg $\iff p \in (-1,1)$

Důkaz tvrzení 3.25: Fix $\varepsilon > 0$. BÚNO: $\mu(E)>0$ (jinak $\int_E f_j = 0$ pro všechna j a je to triviální)

Víme: $\exists j_0 \ \forall j \geq j_0 \ \forall x \in E \ |f_j(x) - f(x)| < \varepsilon / \mu(E)$

Pak pro $j \geq j_0$:

$$\left|\int_E f_j - \int_E f\right| = \left|\int_E f_j - f\right| \leq \int_E |f_j - f| \, d\mu$$

$$\leq \underbrace{\sup_{x\in E} |(f_j - f)(x)|}_{\leq \frac{\varepsilon}{\mu(E)}} \mu(E) \leq \varepsilon$$

Tedy $\lim_j \int_E f_j = \int_E f$ $\boxtimes$

- Příklad — Léviho věta bez p.p. $\int f_1 > -\infty$ neplatí:

$f_n(x) = \frac{1}{n}$, $x \in (0,\infty)$. Pak $f_n(x) \to 0$, $x \in (0,\infty)$,

$f_1 \geq f_2 \geq f_3 \geq \dots$

Ale $\int_0^\infty f_n(x)\,dx = +\infty \not\to \int_0^\infty 0 \, dx$

Příklady na prohození limity a $\int$

(a) $\lim_{n\to\infty} \int_0^1 \frac{x^n}{n} dx$

1. ZP: spočítat

$$= \lim_n \left[\frac{x^{n+1}}{(n+1)n}\right]_0^1 = \lim_n \frac{1}{n(n+1)} = 0.$$

2. ZP: LEVIHO VĚTA

$f_n(x) = \frac{x^n}{n}$, pak • $f_1(x) \geq f_2(x) \geq f_3(x) \geq \dots$

• $\int_0^1 f_1(x)\,dx = \frac{1}{2}$

• $f_n(x) \longrightarrow 0$ tedy $q^n \longrightarrow 0$ pro $|q| < 1$

=> dle Leviho věty:

$$\lim_n \int_0^1 f_n(x) = \int_0^1 \lim f_n(x) = \int_0^1 0 = 0.$$

3. ZP: LEBESGUE:

Máme • $|f_n(x)| \leq 1$, $n \in \mathbb{N}$, $x \in (0,1)$

• $\int_0^1 1$ je konečný

$\overset{\text{Lebesgue}}{\Longrightarrow}$ $$\lim_n \int_0^1 f_n(x) = \int_0^1 \lim_n f_n(x) = \int_0^1 0 = 0$$

4. ZP: $\rightrightarrows$ KCE:

Máme • $\lambda((0,1)) = 1 < \infty$

• $f_n \rightrightarrows 0$

┌ • $|f_n(x)| \leq \frac{1}{n} \longrightarrow 0$, tedy $\sigma_n = \max_{x \in (0,1)} |f_n(x)| \leq \frac{1}{n}$

a $\sigma_n \to 0$ $\underset{\text{TVRZENÍ 2.2}}{\Longrightarrow}$ $f_n \rightrightarrows 0$ ┘

DLE T 3.25 $\Longrightarrow$ $$\lim_n \int_0^1 f_n(x) = \int_0^1 0 = 0.$$

(b) $$\lim_{n \to \infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx$$

┌ 1. ZP: SPOČÍTAT:

$$= \left| \begin{array}{l} t = 1+n^2x^2 \\ dt = 2n^2x\,dx \end{array} \right| = \frac{1}{2n} \int_1^{1+n^2} \frac{1}{t}\,dt = \frac{\log(1+n^2)}{2n} \xrightarrow{\text{R.Š.}} 0$$

2. ZP: $f_n(x) = \frac{nx}{1+n^2x^2}$, pak $f_n(x) \longrightarrow 0$ pro $x \in (0,1)$

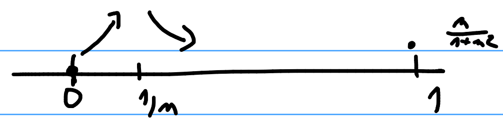
Máme:

$$\sup_{x\in(0,1)} f_n(x) = f_n\left(\tfrac{1}{n}\right) = 1/2$$

$$f_n'(x) = \frac{n(1+n^2x^2) - nx(2n^2x)}{(1+n^2x^2)^2} = \frac{\overset{>0}{n}(1-n^2x^2)}{(\dots)^2}$$

tedy $f_n'(x) = 0 \iff x = \frac{1}{n}$

dále:



Tedy • $|f_n(x)| \le 1/2, \quad n \in \mathbb{N}$

($1/2$ je tedy integrovatelná majoranta na $(0,1)$)

Lebesgue $\Rightarrow$ $$\lim_{n\to\infty} \int_0^1 f_n(x) = \int_0^1 \lim_n f_n(x) = \int_0^1 0 = \underline{0}.$$

(c)

$$\lim_{n\to\infty} \int_0^1 \frac{n^{3/2}x}{1+n^2x^2}\,dx$$

1. ZP: SPOČÍTAT:

$$= \left| \begin{matrix} t = 1+n^2x^2 \\ dt = 2n^2x\,dx \end{matrix} \right| = \frac{1}{2\sqrt{n}} \int_1^{1+n^2} \frac{1}{t}\,dt = \frac{\lg(1+n^2)}{2\sqrt{n}} \to 0$$

2. ZP: SPOČÍTAT:

• $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2} \xrightarrow[n\to\infty]{} 0, \quad x \in (0,1)$

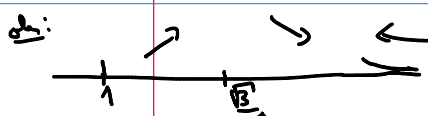
$$\sup_{n\in\mathbb{N}} |f_n(x)| \le \sup_{n\in[1,\infty)} \frac{n^{3/2}x}{1+n^2x^2} = \circledast$$

PRŮBĚH FCE $[1,\infty) \ni n \longmapsto \frac{n^{3/2}x}{1+n^2x^2}$, OZNAČÍME JI $h_x(n)$

$$h_x'(n) = \frac{\frac{3}{2}\sqrt{n}\,x(1+n^2x^2) - n^{3/2}x\,(2nx^2)}{(\dots)^2} = \frac{\overset{>0}{\sqrt{n}\,x}\left(3/2 - 1/2\, n^2x^2\right)}{(\dots)^2}$$

$$\left(= 0 \iff n^2 = \frac{3}{x^2}\right)$$

dle:



$$\Rightarrow \sup_{n\in[1,\infty)} h_x(n) = h_x\left(\frac{\sqrt{3}}{x}\right) =$$

$$(*) = \frac{3^{3/4}}{4\sqrt{x}} =: g(x)$$

Zároveň $\int_0^1 g(x) \in \mathbb{R}$ $\left[\text{protože } \int_0^1 \frac{1}{\sqrt{x}}\, dx \in \mathbb{R}\right]$

$$\overset{\text{LEBESGUE}}{\Longrightarrow} \lim_n \int_0^1 \lim f_n(x) = \int_0^1 \lim_n f_n = \int_0^1 0 = \underline{\underline{0}}$$

(d) $\displaystyle \lim_{a\to\infty} \int_0^{\infty} e^{-ax^2}$

POUŽIJEME HEINEHO VĚTU:

Zvolme $a_n \to \infty$. Chceme spočítat

$$\lim_{n\to\infty} \int_0^{\infty} e^{-a_n x^2}\, dx$$

Máme

- $\left| e^{-a_n x^2} \right| \overset{a_n \geq 1 \text{ (BÚNO, PROTOŽE } a_n \to \infty)}{\leq} e^{-x^2}$
- Zároveň $\int_0^{\infty} e^{-x^2} dx \in \mathbb{R}$

Protože $e^{-x^2} \in \mathcal{C}([0,\infty))$, zároveň $\exists k>0\ \forall x>k: \frac{1}{e^{x^2}} \leq \frac{1}{x^2}$

$\overset{\text{S.K. + FAKT ŽE } \int_k^\infty \frac{1}{x^2} dx \in \mathbb{R}}{\Longrightarrow} \int_k^{\infty} e^{-x^2} dx \in \mathbb{R}$

$$\overset{\text{Lebesgue}}{\Longrightarrow} \lim_{n\to\infty} \int_0^{\infty} e^{-a_n x^2} dx = \int_0^{\infty} \lim_{n\to\infty} e^{-a_n x^2} dx = \int_0^{\infty} 0 = \underline{\underline{0}}.$$

Dle Heineho věty, $\displaystyle \lim_{a\to\infty} \int e^{-ax^2} dx = \underline{\underline{0}}.$

SUMA A INTEGRÁL

DŮSL 3.29:

(i) $f_j = a\, q^j$, $|q| < 1$, $\int \frac{a}{1-q}$ K.

Dk: $\sum f_j(x)$ je konv. (jde o součet geom. řady)

Pro $x \in E$ máme

$$\left|\sum_{j=0}^{N} f_j(x)\right| = |a(x)| \left| \overbrace{\frac{1-q^{N+1}(x)}{1-q(x)}}^{\le 2} \right| \le \frac{|2a(x)|}{|1-q(x)|}, \quad N \in \mathbb{N}$$

(částeční součet geom. řady: $\sum_{k=0}^{N} q^k = \frac{1-q^{N+1}}{1-q}$)

Tedy $\left|\frac{2a}{1-q}\right|$ je integrovatelná majoranta

$\overset{3.28}{\Longrightarrow}$ lze prohodit sumu a $\int$ ■

(ii) $\sum \int |f_j| < \infty$ nebo $\int \sum |f_j| < \infty$

Dk: Dle Leviho věty, máme $\int \sum |f_j| = \sum \int |f_j|$

(protože posl. $\left(\sum_{j=0}^{N} |f_j(x)|\right)_{N \in \mathbb{N}}$ je ~~monotónní~~ neklesající)

Tedy, $\sum \int |f_j| < \infty \iff \int \sum |f_j| < \infty$

Navíc, pokud $\int \sum |f_j| < \infty$, pak

- $\sum |f_j| < \infty$ s.v.

  (protože platí, že $\mu(E) > 0$, $f = \infty$ na E, pak $\int_E f = +\infty$)

- $g = \sum_{j=0}^{\infty} |f_j|$ je integrovatelná majoranta

$\overset{3.28}{\Longrightarrow}$ lze prohodit $\sum$ a $\int$ ■

(iii) $f_j = (-1)^j h_j$, $h_j \searrow 0$, $h_1 \in L^1$

Γ Dk: $\cdot \sum_{j=0}^{\infty} f_j(x)$ je konvergentní dle Leibnizova krit.

Zároveň pro $N \in \mathbb{N}$:

$$\left|\sum_{j=0}^{N} f_j(x)\right| = \left| \overbrace{h_1(x) - h_2(x)}^{\geq 0} + \overbrace{h_3(x) - h_4(x)}^{\geq 0} + h_5(x) \ldots \right|$$

$$= h_1(x) \underbrace{- h_2(x) + h_3(x)}_{\leq 0} \underbrace{- h_4(x) + h_5(x)}_{\leq 0} \ldots \leq h_1(x)$$

$\Rightarrow$ tedy h_1 je integrovatelná majoranta část. součtů

$\overset{3.28}{\Longrightarrow}$ lze prohodit $\sum$ a $\int$ ⊠ $\lrcorner$

PŘÍKLADY

$$\int_0^1 \underbrace{\frac{\log(1-x)}{x}}_{=: f(x)} dx$$

Γ VÍME: $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, $x \in (-1, 1]$

(NEBO TO SOUČTEM: $\log(1+x)' = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$ ... A ŘADU ZINTEGRUJEME ČLEN PO ČLENU ...)

$$\Rightarrow f(x) = -\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \quad x \in (0,1)$$

kde $\sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} dx = \sum_{n=1}^{\infty} \left[\frac{x^n}{n^2}\right]_0^1 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ ✓ LSK o $\sum \frac{1}{n^2}$

$$\overset{3.29\,(iii)}{\Longrightarrow} \int_0^1 f(x)\,dx = -\sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} dx = \underline{\underline{-\sum_{n=1}^{\infty} \frac{1}{n^2}}}$$

$$\int_0^1 \underbrace{\frac{\log(1+x)}{x}}_{=: f(x)} dx$$

Γ VÍME: $\log(1+x) = \sum_{n=1}^{\infty} \frac{-(-x)^n}{n}$, tedy $f(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n+1}$

$$\parallel$$

$$\sum_{n=1}^{\infty} \frac{(-1)(-1)^n x^n}{n}$$

S použitím 3.29 (ii): Máme $\sum_{n=0}^{\infty} \int_0^1 \left| \frac{(-x)^n}{n+1} \right| dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{n+1} dx$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty$$

$$\overset{3.29(ii)}{\Longrightarrow} \quad \underline{\underline{\int f(x)\, dx}} = \sum_{n=0}^{\infty} \int_0^1 \frac{(-x)^n}{n+1} dx = \underline{\underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}}}$$

S použitím 3.29 (iii):

$$f(x) = \sum (-1)^n h_n(x), \quad \text{kde} \quad h_n(x) = \frac{x^n}{n+1}.$$

Pak $h_n(x)$ je klesající pro každé x, $\int h_1(x) < \infty$

$\overset{3.29(iii)}{\Longrightarrow}$ lze prohodit sumu a $\int \dots$ (Tj. platí)

$$\int_0^{\infty} \overbrace{\frac{x}{e^x - 1}}^{=: f(x) \geq 0} dx$$

$$\bullet\ f(x) = \frac{x}{e^x} \frac{1}{1 - e^{-x}} \overset{\substack{\text{pro} \\ x>0}}{=} \frac{x}{e^x} \sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} x\, e^{-(n+1)x}$$

$$\bullet\ \sum_{n=0}^{\infty} \int_0^{\infty} x\, e^{-(n+1)x} dx = \left| \begin{array}{ll} \text{per partes:} & \\ u(x) = x & v'(x) = e^{-(n+1)x} \\ u'(x) = 1 & v(x) = \frac{e^{-(n+1)x}}{-(n+1)} \end{array} \right|$$

$$= \sum_{n=0}^{\infty} \left(\underbrace{\left[x \frac{e^{-(n+1)x}}{-(n+1)} \right]_0^{\infty}}_{=0} + \int_0^{\infty} \frac{e^{-(n+1)x}}{n+1} dx \right)$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{-(n+1)x}}{-(n+1)^2} \right]_0^{\infty} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty$$

$$\overset{3.29(ii)}{\Longrightarrow} \quad \underline{\underline{\int_0^{\infty} f(x)\, dx}} = \sum_{n=0}^{\infty} \int_0^{\infty} x\, e^{-(n+1)x} dx = \underline{\underline{\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}}}.$$

$$\int_0^{\infty} e^{-x} \cos(\sqrt{x}) dx$$

Víme: $\cos y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!}, \ y \in \mathbb{R}$

$\bullet$ Tedy $f(x) = \sum_{n=0}^{\infty} (-1)^n e^{-x} \frac{x^n}{(2n)!}, \quad x > 0$

NA DLUH (PER PARTES.... $I_n = n\, I_{n-1}$, $I_0 = 1$)

$$\bullet \sum_{n=0}^{\infty} \frac{1}{(2n)!} \underbrace{\int_0^{\infty} x^n e^{-x}\, dx}_{= I_n} \overset{\downarrow}{=} \sum_{n=0}^{\infty} \underbrace{\frac{n!}{(2n)!}}_{=: a_n}$$

ZKUSME APLIKOVAT PODÍL KRIT:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+2)!} \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+1)(2n+2)} = 0 < 1$$

podíl krit.

$\Rightarrow$ řada $\sum a_n < \infty$

$$\overset{3.29\,(\dots)}{\Longrightarrow} \underline{\underline{\int_0^{\infty} f(x)\, dx}} = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{x^n e^{-x}}{(2n)!}\, dx = \underline{\underline{\sum_{n=0}^{\infty} \frac{n!}{(2n)!}}}.$$

//

$$\int_0^1 \underbrace{\frac{x^{p-1}}{1+x^q}}_{=: f(x)}\, dx$$

Γ. $f(x) = x^{p-1} \frac{1}{1+x^q} = x^{p-1} \sum_{n=0}^{\infty} (-x^q)^n = \sum_{n=0}^{\infty} (-1)^n x^{p-1+nq}$

$= \sum_{n=0}^{\infty} (-1)^n h_n(x)$, kde $h_n(x) = x^{p-1+nq}$

tj. $h_n(x) \to 0$ pro $x \in (0,1)$, $h_1 \geq h_2 \geq h_3 \geq h_4 \geq \dots$

$$\int_0^1 h_1(x)\, dx = \int_0^1 x^{p-1+q}\, dx \in \mathbb{R}$$

$\downarrow$

$p-1+q > -1$

$$\overset{3.29\,(\dots)}{\Longrightarrow} \underline{\underline{\int_0^1 f(x)\, dx}} = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{p-1+nq}\, dx = \underline{\sum_{n=0}^{\infty} \frac{(-1)^n}{p+nq}}.$$

//

$$\int_0^{\infty} \underbrace{\frac{1}{e^{8x}+1}}_{=: f(x)}\, dx$$

Γ. $f(x) = \frac{1}{e^{8x}} \frac{1}{1+e^{-8x}} = \sum_{n=0}^{\infty} (-1)^n e^{-8xn}\, e^{-8x}$

$= \sum_{n=0}^{\infty} (-1)^n e^{-8(n+1)x}$

$$\left(\bullet \sum_{n=0}^{\infty} \int_0^{\infty} e^{-8(n+1)x}\, dx = \sum_{n=0}^{\infty} \left[\frac{e^{-8(n+1)x}}{-8(n+1)} \right]_0^{\infty} \right.$$

$$= \sum_{n=0}^{\infty} \frac{1}{8(n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n h_n(x) \quad , \text{ lde } \quad h_n(x) = e^{-8(n+1)x} \xrightarrow[n\to\infty]{} 0 \;, x \in [0,\infty)$$

$$h_1 \geq h_2 \geq h_3 \geq \dots$$

$$\int h_1(x) = \int_0^{\infty} e^{-16x}\, dx = \left[\frac{e^{-16x}}{-16} \right]_0^{\infty} = \frac{1}{16} \in \mathbb{R}$$

3.29 (i+ii)

$$\Rightarrow \quad \underline{\underline{\int_0^{\infty} f(x)\, dx}} = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} e^{-8(n+1)x}\, dx = \underline{\underline{\sum_{n=0}^{\infty} (-1)^n \frac{1}{8+8n}}}$$

$$= \frac{1}{8} \int_0^1 \frac{x^0}{1+x}\, dx = \frac{1}{8} \Big[\log(1+x) \Big]_0^1$$

$$= \underline{\underline{\frac{\log 2}{8}}}$$

INTEGRÁL ZÁVISLÝ NA PARAMETRU

MOTIVACE:

Máme $\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} \, dx$

- $\Gamma(s) \in \mathbb{R}$ pro $s>0$ (viz. př. 7 VII. 1-g)
- $\Gamma(s) > 0$ pro $s>0$ (integrál z kladné funkce)
- $\Gamma(1) = \int_0^\infty e^{-x} = \left[-e^{-x}\right]_0^\infty = \underline{\underline{1}}.$
- $\underline{\underline{\Gamma(s+1)}} = \int_0^\infty e^{-x} x^s \, dx = \left| \begin{matrix} \text{PER PARTES:} & \\ u = x^s & v' = e^{-x} \\ u' = s x^{s-1} & v = -e^{-x} \end{matrix} \right|$

$$= \underbrace{\left[-e^{-x} x^s\right]_0^\infty}_{=0-0=0} + \int_0^\infty s e^{-x} x^{s-1} \, dx = \underline{\underline{s\,\Gamma(s)}}$$

- Speciálně: $\forall n \in \mathbb{N}$: $\underline{\underline{\Gamma(n+1)}} = n\Gamma(n) = n(n-1)\Gamma(n-1)$

$$= \dots = \underline{\underline{n!}}.$$

$\Rightarrow \Gamma$ je ZOBECNĚNÍ POJMU FAKTORIÁL

POZN. Právě jsme dokázali Tvrzení 3.32(i),(ii).

DK VĚTY 3.30:

DOKÁŽEME PRO $n=1$.

ZVOLME $a_j \to a$, $a_j \in U$ pro $j \in \mathbb{N}$

PAK

$$\lim_{j\to\infty} F(a_j) = \lim_{j\to\infty} \int_E \underbrace{f(a_j,x)\,d\mu(x)}_{\text{MĚŘ. FCE dle (ii)}} = \int_E \lim_{j\to\infty} f(a_j,x)\, d\mu(x) \overset{\substack{\text{(dle (i))} \\ \text{SPOJITOST } f(\cdot,x)}}{=} \int_E f(a,x)\, d\mu(x)$$

$$\int_E f(a,x)\, d\mu(x) = F(a)$$

DLE (iii) EX. INTEGR. MAJORANTA $|f(a_j,x)| \leq g(x)$, $j \in \mathbb{N}$

$\Rightarrow$ POUŽIJEME LEBESGUEOVU VĚTU

$\overset{\text{HEINE}}{\Longrightarrow}$ F je SPOJ. v bodě a $\boxtimes$

• Γ je spojitá na $(0,\infty)$: $\quad \Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx$

Zvol $s_0 \in (0,\infty)$. Najdeme $0 < p < s_0 < q < \infty$. Pak

• $\forall s>0$: $\quad x \longmapsto e^{-x} x^{s-1}$ měřitelná (je dokonce spoj.)

• $\forall x \in (0,\infty)$: $\quad s \longmapsto e^{-x} x^{s-1}$ spojitá

• $\forall s \in [p,q]$: $\quad \left| e^{-x} x^{s-1} \right| \leq e^{-x} \left(x^{p-1} + x^{q-1} \right) =: g(x)$

$g \in L^1((0,\infty))$ ~~bo~~ protože $\int_0^\infty g(x) = \Gamma(p) + \Gamma(q) \in \mathbb{R}$

$\overset{3.30}{\Longrightarrow}$ Γ je spojitá v (p,q), speciálně je spoj. v bodě s_0

s_0 byl libovolný $\Rightarrow$ Γ spoj. v $(0,\infty)$.

DISKUSE K VĚTĚ 3.31 :

• BEZ P.P. (ii) VĚTA NEPLATÍ :

PŘ.: $\quad F(\lambda) = \int_0^1 \frac{x^\lambda}{\lg x}\, dx \quad , \quad \lambda < -1$

Pak $F(\lambda) = -\infty$ pro $\lambda < -1$.

ale $\quad \int_0^1 \frac{\partial}{\partial \lambda} \left(\frac{x^\lambda}{\lg x} \right) dx = \int_0^1 x^\lambda\, dx = \frac{1}{\lambda+1}$

tedy $\quad F'(\lambda) \neq \int_0^1 \frac{\partial}{\partial\lambda} \left(\frac{x^\lambda}{\lg x} \right) dx$.

• BEZ P.P. (iii) VĚTA NEPLATÍ :

PŘ.: $\quad F(\lambda) = \int_{-\infty}^{\infty} \frac{\cos \lambda x}{1+x^2}\, dx \quad , \quad \lambda \in \mathbb{R}$

Máme $\quad \int_{-\infty}^{\infty} \frac{\partial}{\partial\lambda} \left(\frac{\cos \lambda x}{1+x^2} \right) dx = - \underbrace{\int_{-\infty}^{\infty} \frac{x \sin(\lambda x)}{1+x^2}\, dx}_{\in \mathbb{R} \iff \lambda = 0}$

$$\bullet\; F(\lambda) = \left| \begin{array}{ll} \text{PER PARTES} & \\ u = \frac{1}{1+x^2} & v' = \cos(\lambda x) \\ u' = \frac{-2x}{(1+x^2)^2} & v = \frac{+\sin(\lambda x)}{\lambda} \end{array} \right|$$

$$= \underbrace{\left[\frac{\sin(\lambda x)}{\lambda(1+x^2)} \right]_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} \frac{2x \sin(\lambda x)}{\lambda (1+x^2)^2}\, dx$$

Zdůvodnění: $\left|\frac{\partial}{\partial \lambda}\left(\frac{2x \sin(\lambda x)}{\lambda (1+x^2)^2}\right)\right| = \left|\frac{2x}{\underbrace{(1+x^2)^2}_{\geq 1+x^4}}\right| \left|\frac{\lambda^2 \cos(\lambda x) - \sin(\lambda x)}{\lambda^2}\right|$

$$\leq \left|\frac{2x}{1+x^4}\right| \left|\cos(\lambda x) - \frac{\sin(\lambda x)}{\lambda^2}\right|$$

$$\leq \left|\frac{2x}{1+x^4}\right| \left(1 + \frac{1}{\lambda^2}\right) \leq \left|\frac{2x}{1+x^4}\right| \left(1 + \frac{1}{\delta^2}\right)$$

pro $\lambda \in \mathbb{R} \setminus (-\delta, \delta)$

$\Rightarrow$ pro $\lambda \neq 0$ máme integrovatelnou majorantu, a proto

$$F'(\lambda) = \underbrace{\int_{-\infty}^{\infty} \frac{2x}{(1+x^2)^2} \frac{\lambda^2 \cos(\lambda x) - \sin(\lambda x)}{\lambda^2}\, dx}_{\in \mathbb{R}}, \quad \lambda \in \mathbb{R} \setminus \{0\}$$

DERIVACE FUNKCE Γ:

$$\left|\frac{\partial^{(k)}}{\partial s}\left(e^{-x} x^{s-1}\right)\right| = \left|e^{-x} x^{s-1} (\lg x)^k\right| \leq$$

$$\leq e^{-x} |(\lg x)^k| \left(x^{p-1} + x^{q-1}\right) \in L^1((0,\infty))$$

pro $s \in (p, q)$ — dle VII.1.g [Př.]

$\Rightarrow \forall\, 0 < p < q < \infty$:

$$\forall s \in (p,q): \quad \Gamma^{(k)}(s) = \int_0^{\infty} e^{-x} x^{s-1} (\lg x)^k \, dx \in \mathbb{R}$$

$$\Rightarrow \Gamma \in \mathcal{C}^{(k)}((0,\infty)), \quad k \in \mathbb{N}$$

Pozn.: PRÁVĚ JSME DOKÁZALI TVRZENÍ 3.32 (iii)

DALŠÍ PŘÍKLADY

$$F(a) = \int_0^{\infty} \frac{e^{-ax}}{1+x^2}\, dx$$

- URČÍME DEF. OBOR: PRO $a \geq 0$: $\left|\frac{e^{-ax}}{1+x^2}\right| \leq \frac{1}{1+x^2} \in L^1((0,\infty))$

$\Rightarrow F(a) \in \mathbb{R}$ pro $a \geq 0$

PRO $a < 0$: máme $\lim_{x\to\infty} \frac{\frac{e^{-ax}}{1+x^2}}{1/x} = \lim_{x\to\infty} \underbrace{\frac{e^{|a|x}}{x}}_{\text{z.l.} \to \infty} \underbrace{\frac{x^2}{1+x^2}}_{\to 1} = +\infty$

neboť $\exists k>0\ \forall x>k$: $\frac{e^{-ax}}{1+x^2} \geq \frac{1}{x} \notin L^1((0,\infty))$

$\overset{s.k.}{\Rightarrow} F(a) \notin \mathbb{R}$.

Celkem: def. obor je $[0,\infty)$

Navíc:
$$\left|\frac{e^{-ax}}{1+x^2}\right| \leq \frac{1}{1+x^2} \in L^1((0,\infty)), \quad a \geq 0$$

$\overset{3.30}{\Rightarrow}$ $F(a)$ je spojitá na $[0,\infty)$.

(PRO SPOJITOST V NULE POUŽIJEME HEINEHO VĚTU)

$$F(a) = \int_0^{\infty} e^{-ax}\, dx$$

- pro $a<0$ je $F(a) = +\infty$ (neboť $\lim_{x\to\infty} e^{-ax} = +\infty$,

neboť $\exists k>0\ \forall x>k$: $e^{-ax} \geq 1 \notin L^1((k,\infty))$

$\overset{s.k.}{\Rightarrow}$ $F(a) = +\infty$)

- $F(0) = +\infty$.
- pro $a>0$:

$\exists k>0\ \forall x\geq k$: $\left|e^{-ax}\right| \leq \frac{1}{x^2} \in L^1((k,\infty))$ $\overset{s.k.}{\Rightarrow}$ $F(a) \in \mathbb{R}$

Celkem: def. obor fce F je $(0,\infty)$

Pro $a_0>0$ najdeme $p>0$ tž. $a_0>p>0$

pak
$$\left|e^{-ax}\right| \leq e^{-px} \in L^1((0,\infty)), \quad a>p$$

$\overset{3.30}{\Rightarrow}$ F je spojitá v (p,∞), speciálně v bodě a_0

Protože $a_0>0$ bylo libovolné, F je spoj. na $(0,\infty)$.

$$F(a) = \int_0^1 \underbrace{\log(x^2+a^2)}_{f(a,x)}\, dx$$

• Def. obor: $x \longmapsto f(a,x) \in \mathcal{C}((0,1])$, je třeba tedy zjistit kai „v 0"

pro $a \neq 0$: $x \mapsto f(a,x) \in \mathcal{C}([0,1])$, tedy integrál konverguje

pro $a = 0$: ~~$\int_0^1 \log(x^2)\,dx$ ... $f(0,x) \in \mathcal{C}((0,1])$~~

$$F(0) = \int_0^1 \log(x^2)\,dx$$

$x^a \log x \xrightarrow[x \to 0+]{} 0$ ⟵ Víme: $\exists \delta > 0\ \forall x \in (0,\delta):\ \log(x^2) \leq \frac{1}{\sqrt{x}} \in L^1((0,\delta))$

$$\overset{\text{S.K.}}{\Longrightarrow} F(0) \in \mathbb{R}$$

Celkem: $F(a) \in \mathbb{R}$ pro $a \in \mathbb{R}$

• F je SUDÁ $\Rightarrow$ stačí zjistit spojitost pro $a \geq 0$

$$\left|\log(x^2+a^2)\right| \underset{\substack{a \in [p,q] \\ p,q > 0}}{\leq} \underbrace{\left|\log(x^2+p^2)\right| + \left|\log(x^2+q^2)\right|}_{\in L^1((0,1))}, \quad x \in (0,1)$$

$$\left(\text{protože } \mathbb{R} \ni F(p) = \int_0^1 \log(x^2+p^2)\,dx \quad \text{a} \quad \mathbb{R} \ni F(q) = \int_0^1 \log(x^2+q^2)\,dx\right)$$

$\Rightarrow F$ je spojitá na $[p,q]$ pro každé $0 < p < q < \infty$

$\Rightarrow F$ je spojitá na $(0,\infty)$, tedy na $\mathbb{R} \setminus \{0\}$

Pozn.: SPOJITOST V NULE ZASE POMOCÍ HEINEHO VĚTY ... (VYNECHÁME)

INTEGRÁL ZÁVISLÝ NA PARAMETRU

DŮKAZ 3.32: (i) – (iii) : BYLO MINULE

(iv): DOKONCE VÍME:

$$\Gamma^{(k)}(s) = \int_0^{\infty} e^{-x} \, x^{s-1} \, (\log x)^k \, dx$$

Tedy, $\Gamma^{(k)}(s) > 0$ pro k sudé a $s > 0$

$\Rightarrow \Gamma$ je RYZE KONVEXNÍ na $(0, \infty)$

(v): • $\lim\limits_{s \to 0^+} \Gamma(s) \underset{\text{(iii)}}{=} \lim\limits_{s \to 0^+} \dfrac{\Gamma(s+1)}{s} \underset{\Gamma \text{ spojitá}, \ \Gamma(1)=1}{=} \text{"} \dfrac{1}{0^+} \text{"} = \underline{\underline{+\infty}}.$

• Protože $\Gamma''(s) > 0$ pro $s > 0$, $\Gamma'(s)$ roste. Dále

LAGRANGE VĚTA O STŘEDNÍ HODNOTĚ $\rightarrow$ $\exists \xi \in (1,2) : \dfrac{\Gamma(2) - \Gamma(1)}{2-1} = \Gamma'(\xi)$

$\dfrac{\Gamma(2) - \Gamma(1)}{2-1} = 0 \quad (\Gamma(1) = 1, \ \Gamma(1+1) = 1!)$

$\Rightarrow \Gamma'(s) > 0$ pro $s > 2$

$\Rightarrow \Gamma(s)$ roste na $(2, \infty) \underset{\text{MONOTONIE } \Gamma}{\Rightarrow} \lim\limits_{s \to \infty} \Gamma(s)$ existuje

Zároveň (Heine, $s_n = n+1$)

$\lim\limits_{s \to \infty} \Gamma(s) = \lim\limits_{n \to \infty} \Gamma(n+1) = \lim\limits_{n \to \infty} n! = \underline{\underline{+\infty}}.$

(vi):

Použijme:

(*) $\forall s > 0$: $\underline{\Gamma(s)} = \left| \begin{matrix} x = r^2 \\ dx = 2r\,dr \end{matrix} \right| = \displaystyle\int_0^{\infty} e^{-r^2} \, r^{2(s-1)} \, 2r \, dr$

$$= 2 \int_0^{\infty} e^{-r^2} \, r^{2s-1} \, dr$$

Speciálně, $\Gamma(1/2) = 2\int_0^\infty e^{-r^2}\,dr = \underline{\underline{\sqrt{\pi}}}$.

$\boxtimes$

$\boxed{\text{FUNKCE BETA}}$:

DŮKAZ 3.33: (i): VÍME, ŽE $B(p,q) \in \mathbb{R}$ (VIZ. PŘÍKLADY NA KONV. INTEGRÁLU)

ZBYTEK LEHKÝ

(ii):

$$p\,B(p,q+1) = \int_0^1 p\,x^{p-1}(1-x)^q\,dx = \left|\begin{matrix}\text{PER PARTES:} & \\ u' = p\,x^{p-1} & v = (1-x)^q \\ u = x^p & v' = -q(1-x)^{q-1}\end{matrix}\right|$$

$$= \underbrace{\left[x^p(1-x)^q\right]_0^1}_{=0} + \int_0^1 x^p\,q\,(1-x)^{q-1}\,dx$$

$$= q\,B(p+1,q),$$

(iii):

$\forall p,q>0$:

(**)

$$B(p,q) = \left|\begin{matrix} x = \cos^2\alpha \\ dx = -2\cos\alpha\sin\alpha\,d\alpha \\ \alpha\in(0,\frac{\pi}{2})\end{matrix}\right| = \int_0^{\frac{\pi}{2}} \cos^{2(p-1)}\alpha\,\sin^{2(q-1)}\alpha\; 2\cos\alpha\sin\alpha\,d\alpha$$

$$= 2\int_0^{\pi/2} \cos^{2p-1}\alpha\,\sin^{2q-1}\alpha\,d\alpha$$

$\Rightarrow$ Tedy, pro $p,q>0$:

$$\Gamma(p)\Gamma(q) \overset{(*)}{=} 4\int_0^\infty\int_0^\infty e^{-x^2-y^2}\,x^{2p-1}\,y^{2q-1}\,dx\,dy$$

$$\underset{\substack{\text{POLÁRNÍ}\\\text{SOUŘADNICE}}}{=} \text{+FUBINI} \quad 4\int_0^\infty\int_0^{\pi/2} r\,e^{-r^2}(r\cos\alpha)^{2p-1}(r\sin\alpha)^{2q-1}\,d\alpha\,dr$$

$$= \left(2\int_0^\infty e^{-r^2}\,r^{2p+2q-1}\,dr\right)\left(2\int_0^{\pi/2}(\cos\alpha)^{2p-1}(\sin\alpha)^{2q-1}\,d\alpha\right)$$

$$\overset{(*)+(**)}{=} \Gamma(p+q)\cdot B(p,q)$$

(iv): PLYNE HNED Z (iii)

(ad) : $\underline{B(1-s,s)} \overset{(viz)}{=} \dfrac{\Gamma(1-s)\Gamma(s)}{\Gamma(1)} = \underline{\Gamma(1-s)\,\Gamma(s)}$

DŮKAZ ŽE $B(1-s,s) = \dfrac{\pi}{\sin(\pi s)}$, $s \in (0,1)$ vynechán (složitý)

$\boxtimes$

PŘÍKLADY

2. $$F(a) = \int_0^\infty \frac{e^{-ax^2}}{1+x^2}\,dx$$

- Def. obor: • $\forall a \geq 0$: $\left|\dfrac{e^{-ax^2}}{1+x^2}\right| \leq \dfrac{1}{1+x^2} \in L^1([0,\infty))$

$\Rightarrow F(a) \in \mathbb{R}$ pro $a \geq 0$ (dle s.k.)

• $\forall a < 0$: máme $\lim\limits_{x\to\infty} \dfrac{e^{-ax^2}}{1+x^2} \overset{\text{z.l. (TŘEBA POMOCÍ L'H...)}}{=} +\infty$

$\Rightarrow \exists K\, \forall x > K$: $\dfrac{e^{-ax^2}}{1+x^2} \geq 1$; zároveň $\int_0^\infty 1 = +\infty$ $\overset{\text{s.k.}}{\Longrightarrow} F(a) = +\infty$ pro $a < 0$

Celkem: def obor je $[0,\infty)$

• Na $(0,\infty)$ je F spojitá (integrovatelná majoranta je $\dfrac{1}{1+x^2}$ ... viz.)

• $\boxed{\lim\limits_{a\to\infty} F(a)}$: Použijeme Heineho:

Zvol $a_n \to \infty$, pak

$\left|\dfrac{e^{-a_n x^2}}{1+x^2}\right| \leq \dfrac{1}{1+x^2} \in L^1([0,\infty))$, $n \in \mathbb{N}$

$$\lim_{n\to\infty} F(a_n) = \lim_{n\to\infty} \int_0^\infty \frac{e^{-a_n x^2}}{1+x^2}\,dx \overset{\text{Lebesgue}}{=} \int_0^\infty \lim_{n\to\infty} \frac{e^{-a_n x^2}}{1+x^2}\,dx$$

$$= \int_0^\infty 0\,dx = \underline{0}$$

Celkem, dle Heineho věty, dostáváme $\underline{\lim\limits_{a\to\infty} F(a) = 0}$

• $\boxed{\lim\limits_{a\to 0^+} F(a)}$: zvol $a_n \searrow 0$. Pak zase dle Leb. věty (viz.), máme $\lim\limits_{n\to\infty} F(a_n) = \int_0^\infty \lim \dfrac{e^{-a_n x^2}}{1+x^2}\,dx = \int_0^\infty \dfrac{1}{1+x^2}\,dx = \dfrac{\pi}{2}$

Zase, dle Heineho věty,

$$\lim_{a\to 0^+} F(a) = \frac{\pi}{2} = \int_0^\infty \frac{1}{1+x^2}\,dx = F(0)$$

$\Rightarrow$ F je zprava spojitá v nule.

• Dále,

$$\left|\frac{\partial}{\partial a}\left(\frac{e^{-ax^2}}{1+x^2}\right)\right| = \left|\frac{-x^2 e^{-ax^2}}{1+x^2}\right| \overset{\text{pro } a\in[p,q]}{\leq} \underbrace{\frac{x^2}{1+x^2}e^{-px^2}}_{\in L^1((0,\infty))}$$

"u ∞": $\displaystyle\lim_{x\to\infty}\frac{\frac{x^2}{1+x^2}e^{-px^2}}{e^{-px^2}} = 1 \in \mathbb{R}\setminus\{0\}$

$\overset{LSK}{\Longrightarrow}$ + FAKT ŽE $\int_0^\infty e^{-px^2}\,dx \in \mathbb{R}$ [protože $\frac{1}{x^2}\in L^1(\text{na }(1,\infty))$

a $\displaystyle\lim_{x\to\infty}\frac{e^{-px^2}}{\frac{1}{x^2}} = 0$]

$\frac{x^2}{1+x^2}e^{-px^2} \in L^1((0,\infty))$ pro $p>0$]

Tedy, pro $0<p<q<\infty$ máme integrovatelnou majorantu na (p,q),

tj.:

$$F'(a) = \int_0^\infty \frac{-x^2 e^{-ax^2}}{1+x^2}\,dx, \quad a\in(p,q)$$

Ale $p,q>0$ byly libovolné, tedy

$$F'(a) = \int_0^\infty \underbrace{\frac{-x^2 e^{-ax^2}}{1+x^2}}_{<0}\,dx, \quad a>0$$

$\Rightarrow$ F je klesající na $[0,\infty)$

• Dále, pro $0<p<q<\infty$ a pro $a\in(p,q)$ máme:

$$\left|\frac{\partial}{\partial a}\left(\frac{-x^2 e^{-ax^2}}{1+x^2}\right)\right| \leq \frac{x^4 e^{-px^2}}{1+x^2} =: g(x)$$

Konvergence $\int g(x)$:

$g(x)\in \mathcal{C}([0,\infty))$, stačí vyšetřit jen "u ∞"

"u ∞": $\displaystyle\lim_{x\to\infty}\frac{g(x)}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{x^6}{\underbrace{(1+x^2)}_{\geq 1}e^{px^2}} = 0$

$\frac{x^6}{(1+x^2)e^{px^2}} \leq \frac{x^6}{e^{px^2}} \xrightarrow{\text{z.l.}} 0$]

$\Rightarrow \exists K\ \forall x>K:\quad g(x) \le \frac{1}{x^2} \in L^1((1,\infty))$

$\overset{S.K.}{\Rightarrow}\ g \in L^1((K,\infty))$ $\quad\Rightarrow\quad g \in L^1((0,\infty))$

$g \in \mathcal{C}([0,K])$

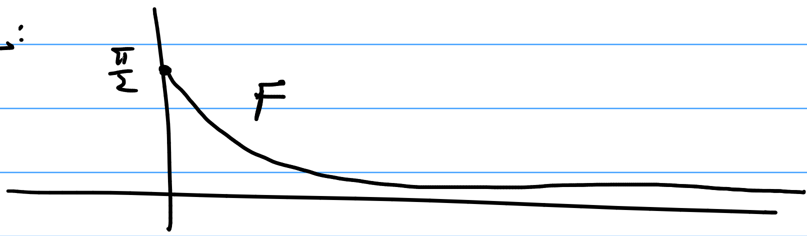
3.31

$\Rightarrow \forall a\in(p,q):\quad F''(a) = \int_0^\infty \underbrace{\frac{x^4 e^{-ax^2}}{1+x^2}}_{>0}\, dx > 0$

(p,q bylo libovolné, tedy $\forall a>0:\quad F''(a) = \int_0^\infty \frac{x^4 e^{-ax^2}}{1+x^2}\, dx > 0$)

$\Rightarrow F$ je RYZE KONVEXNÍ na $(0,\infty)$

náčrtek grafu:



3

$$F(a) = \int_0^\infty \underbrace{\frac{1-e^{-ax}}{xe^x}}_{f_a(x)}\, dx$$

• Def. obor:

$f_a(x) \in \mathcal{C}((0,\infty))$ ... je třeba vyšetřit oba integrály „v 0" a „v ∞"

„v 0": [IDEA: $f_a(x) \approx \frac{ax}{x} \approx a \Rightarrow$ Sbíhá integrál]

$$\lim_{x\to 0+} f_a(x) \overset{AL}{=} 1\cdot \lim_{x\to0+} \frac{-(e^{-ax}-1)}{-ax}\cdot a \overset{AL + VOLSF}{=} -a \cdot \lim_{y\to 0}\frac{e^y-1}{y} = -a \in \mathbb{R}$$

$\Rightarrow$ „v 0" integrál $\int_0^1 f_a(x)\,dx$ konverguje pro $a\in\mathbb{R}$

„v ∞": [IDEA: $\approx \frac{e^{-(a+1)x}}{x}$ ... kde je pro $a+1>0$]

Pro $a>-1$: zvol $a' \in (0,a+1)$. Pak

$$\lim_{x\to\infty}\frac{f_a(x)}{e^{-a'x}} = \lim_{x\to\infty}\frac{(1-e^{-ax})\,e^{a'x}}{xe^x} \overset{AL}{=} 0 - \lim_{x\to\infty}\frac{e^{\overbrace{x(a'-a-1)}^{<0}}}{x} = 0$$

$\Rightarrow \exists K\ \forall x>K\quad f_a(x) \le e^{-a'x} \in L^1((K,\infty)) \Rightarrow f_a(x) \in L^1((K,\infty))$

Pro $a\le -1$: $\quad |f_a(x)| \ge \left|\frac{1-e^x}{xe^x}\right| \notin L^1((1,\infty))$ → PROTOŽE $\lim_{x\to\infty}\left(\frac{1-e^x}{xe^x}\right)/\frac{1}{x} \in \mathbb{R}\setminus\{0\}$

Celkem: $\quad F(a) \in \mathbb{R}$ pro $a>-1$

ZKUSME PROHODIT INTEGRÁL A DERIVACI:

$\to$ dle LSK + FAKTU že $\int_1^{\infty} \frac{1}{x} = +\infty$ dokážeme že $\frac{1-e^x}{x e^x} \notin L^1((1,\infty))$

NĚJAKÉ APLIKACE 1

DOKONČENÍ

VÍME: $$F(a) = \int_0^\infty \frac{1-e^{-ax}}{xe^x}\,dx$$

$$F(a) \in \mathbb{R} \quad \text{pro } a > -1$$

- Máme
$$\left|\frac{\partial}{\partial a}\left(\frac{1-e^{-ax}}{xe^x}\right)\right| = \left|\frac{1}{xe^x}\, x e^{-ax}\right| = e^{-(a+1)x} \le e^{-(p+1)x}$$

pro $a \in [p, \infty)$

Máme $e^{-p'x} \in L^1((0,\infty))$ pro $p' > 0$

(PROTOŽE $\frac{1}{e^{p'x}} \ll \frac{1}{x^2}$ pro $p' > 0$ ... POUŽIJEME S. K.)

Celkem: $\forall p > -1 \quad \forall a \in [p, \infty)$:

$$\left|\frac{\partial}{\partial a}\left(\frac{1-e^{-ax}}{xe^x}\right)\right| \le e^{-(p+1)x} \in L^1((0,\infty))$$

$$\overset{3.31}{\Longrightarrow} \quad F'(a) = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{1-e^{-ax}}{xe^x}\right) dx, \quad a > -1$$

PLATÍ PRO $p > -1$ $\Rightarrow$ TJ. PRO $a > -1$

$$= \int_0^\infty e^{-(a+1)x}\,dx$$

$$= \left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^\infty = 0 + \frac{1}{a+1}$$

$$= \frac{1}{a+1}$$

$\Rightarrow$ Tedy, $F(a) = \log(a+1) + C, \quad a > -1$

(a pro nějaké $C \in \mathbb{R}$)

Zároveň, $F(0) = \int_0^\infty 0\,dx = 0 \Rightarrow C = 0$

Tedy, $$\underline{\underline{F(a) = \log(a+1), \quad a > -1}}.$$

$\boxed{\text{APLIKACE FUNKCÍ } \Gamma \text{ a } B}$

<u>Víme:</u> $$B(p,q) = B(q,p) = 2\int_0^{\pi/2} \sin^{2q-1} x \, \cos^{2p-1} x \, dx$$

(viz. IX.1.a)

$$\underline{B(p,q)} = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx = \left| \begin{array}{l} t = \frac{1}{x} - 1 \\ dt = -\frac{1}{x^2} \, dx \end{array} \right|$$

$$= \underline{\int_0^\infty \frac{t^{q-1}}{(t+1)^{p+q}} \, dt}$$

$\hookrightarrow$ $$x^{p-1} (1-x)^{q-1} x^2 = x^{p+1} (1-x)^{q-1} = \left(\frac{1}{t+1}\right)^{p+1} \left(1 - \frac{1}{t+1}\right)^{q-1}$$

$$= \frac{t^{q-1}}{(t+1)^{p+1} (t+1)^{q-1}}$$

$$\int_0^\infty x^4 e^{-x^2} \, dx = \left| \begin{array}{l} t = x^2 \\ dt = 2x\,dx \end{array} \right| = \int_0^\infty \frac{1}{2} t^{3/2} e^{-t} \, dt$$

$$= \frac{1}{2} \Gamma\left(\frac{5}{2}\right) = \frac{1}{2} \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \underline{\underline{\frac{3}{8}\sqrt{\pi}}}.$$

$$\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx = \frac{1}{2} B\left(\frac{5}{2}, \frac{7}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)}$$

$$2p-1 = 4$$
$$2q-1 = 6$$

$$= \frac{1}{2} \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \left(\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{5!} = \pi \frac{3 \cdot \cancel{1} \cdot \cancel{3} \cdot \cancel{5}}{\cancel{5} \cdot 4 \cdot \cancel{3} \cdot 2 \cdot 2^6} = \underline{\underline{\pi \frac{3}{2^9}}}.$$

$$\int_0^\infty \frac{x^{p-1}}{1+x^n}\,dx = \left| \begin{array}{l} t = x^n \\ dt = n x^{n-1}\,dx \end{array} \right|$$

$$= \int_0^\infty t^{\frac{p-1}{n}} \frac{1}{1+t} \frac{1}{n} t^{\frac{1-n}{n}}\,dt$$

$$= \frac{1}{n}\int_0^\infty t^{\frac{p-n}{n}} \frac{1}{1+t}\,dt$$

$$= \frac{1}{n} B\left(\frac{p}{n}, 1-\frac{p}{n}\right) = \underline{\underline{\frac{1}{n} \frac{\pi}{\sin\left(\pi \frac{p}{n}\right)}}}$$

„$q' = \frac{p}{n}$; $p' = 1-\frac{p}{n}$" pro $p < n$

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx = \int_0^1 x^{1-1} (1-x^4)^{\frac{1}{2}-1}\,dx$$

$$= \left| \begin{array}{l} t = x^4 \\ dt = 4x^3\,dx \end{array} \right| = \int_0^1 \frac{1}{4} t^{-3/4} (1-t)^{1/2-1}\,dt$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4} \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}}{4} \frac{(\Gamma(1/4))^2}{\Gamma(1/4)\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4} (\Gamma(1/4))^2 \frac{1}{\sqrt{2}\,\pi} = \underline{\underline{\frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}}}.$$

$$\Gamma(1/4)\,\Gamma(3/4) = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\,\pi$$

$(p < n)$

$$\int_{-\infty}^\infty \frac{e^{px}}{1+e^{nx}}\,dx = \left| \begin{array}{l} t = e^x \\ dt = e^x\,dx \end{array} \right|$$

$$= \int_0^\infty \frac{t^{p-1}}{1+t^n}\,dt = \underline{\frac{1}{n} \frac{\pi}{\sin\left(\frac{\pi p}{n}\right)}}.$$

úz. 2c

<u>OBJEM n-D KOULE</u>: <u>obsah kruhu</u>: πR^2

<u>objem koule</u>: $\frac{4}{3}\pi R^3$

DŮKAZ 3.34:

1. krok:

$$\lambda^n(B(0,R)) = R^n \cdot \lambda^n(B(0,1))$$

Dk: Pro $A \subseteq \mathbb{R}^n$ máme:

$$\lambda^n(RA) = \int_{\{x \in RA\}} 1 \, d\lambda^n = \left| \lambda = \frac{x}{R} \right| = \int_{\{\lambda \in A\}} R^n \cdot 1 \, d\lambda^n = R^n \lambda^n(A)$$

↳ VĚTA O SUBSTITUCI:

$\varphi: \lambda \mapsto \lambda \cdot R$, PAK $|J_\varphi| = \begin{vmatrix} R & 0 & \cdots & 0 \\ 0 & R & 0 \cdots & 0 \\ & & \ddots & 0 \\ & 0 & & R \end{vmatrix} = R^n$

$\square$

TEDY BÚNO $R=1 \Rightarrow$ 2. KROK:

Máme

$$(\sqrt{\pi})^n \underset{\text{LAPL. INTEGRÁL}}{=} \left(\int_{\mathbb{R}} e^{-x^2} \, dx \right)^n$$

$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-\sum_{i=1}^n x_i^2} \, dx_1 \, dx_2 \cdots dx_n$$

$$\overset{\text{FUBINI}}{=} \int_{\mathbb{R}^n} e^{-\|x\|^2} \, d\lambda^n(x)$$

$\|x\|^2 := \sum x_i^2$

$$= \int_{\mathbb{R}^n} \left(\int_0^{e^{-\|x\|^2}} 1 \, d\lambda \right) d\lambda^n(x)$$

$$\overset{\text{FUBINI}}{=} \int_0^1 \lambda^n\left(B\left(0, \sqrt{\log\left(\tfrac{1}{\lambda}\right)}\right)\right) d\lambda$$

$$\{(x,\lambda);\ x \in \mathbb{R}^n,\ \lambda \in (0, e^{-\|x\|^2})\} = \{(x,\lambda);\ \lambda \in (0,1),\ \|x\| \in [0, \sqrt{\log(\tfrac{1}{\lambda})})\}$$

$\lambda < e^{-\|x\|^2}$
$\log \lambda < -\|x\|^2$
$-\log \lambda > \|x\|^2$
$\sqrt{-\log \lambda} > \|x\|$

$$\overset{\text{1. krok}}{=} \int_0^1 \log\left(\tfrac{1}{\lambda}\right)^{n/2} \lambda^n(B(0,1)) \, d\lambda$$

$$= \lambda^n(B(0,1)) \int_0^1 \log\left(\tfrac{1}{\lambda}\right)^{n/2} d\lambda = \left| \begin{matrix} y = \log \frac{1}{\lambda} = -\log \lambda \\ dy = -\frac{1}{\lambda} \, d\lambda \end{matrix} \right|$$

$$= \text{—〃—} \cdot \int_0^\infty y^{n/2} e^{-y} \, dy = \lambda^n(B(0,1)) \cdot \Gamma\left(\tfrac{n}{2}+1\right)$$

$\Rightarrow$ Tedy $\lambda^n(B(0,1)) = \dfrac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ $\boxtimes$

CHOVÁNÍ FAKTORIÁLU: STIRLINGŮV VZOREC

„1" POLICIE...

Př: $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{(2n)!}{(n!)^2}} = \lim\limits_{n\to\infty} \sqrt[n]{\left(\dfrac{f(n)}{n!}\right)^2} \sqrt[n]{\dfrac{(2n)!}{f(2n)}} \cdot \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{f(2n)}{f(n)^2}}$

$f(n) := \sqrt{2\pi}\,\sqrt{n}\,\left(\dfrac{n}{e}\right)^n$ $\left[\text{PAK } \lim\limits_{n\to\infty} \dfrac{n!}{f(n)} = 1\right]$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{\sqrt{2\pi}\,\sqrt{2n}\,\left(\frac{2n}{e}\right)^{2n}}{2\pi\; n\; \left(\frac{n}{e}\right)^{2n}}}$$

$$\overset{AL}{=} \lim_{n\to\infty} \sqrt[n]{\frac{(2n)^{2n}}{\sqrt{n}\;n^{2n}}} = \lim_{n\to\infty} \frac{(2n)^2}{n^2} \sqrt[n]{\frac{1}{\sqrt{n}}}$$

$\sqrt[n]{c} \to 1$

$$\overset{AL}{=} 4 \lim_{n\to\infty} \frac{1}{\sqrt[n]{\sqrt{n}}} \overset{POLICIE}{=} \underline{\underline{4}}.$$

$1 \le \sqrt[n]{\sqrt{n}} \le \sqrt[n]{n} \to 1$

RN VĚTA

Př: $$\lim_{n\to\infty} \lambda^n(B(0,1)) = \lim_{n\to\infty} \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})} \overset{AL}{=} \lim_{n\to\infty} \frac{\pi^{n/2}}{f(n/2)}$$

[$f(x) := \sqrt{2\pi}\,\sqrt{x}\,\left(\frac{x}{e}\right)^x$, PAK

$$\lim_{x\to\infty} \frac{\Gamma(1+x)}{f(x)} = 1$$

STIRLING VZOREC]

$$\overset{AL}{=} \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{\pi^{n/2}}{\frac{\sqrt{n}}{\sqrt{2}}\left(\frac{n}{2e}\right)^{n/2}} \overset{AL}{=} \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \frac{(2e\pi)^{n/2}}{\sqrt{n}\; n^{n/2}}$$

$$\overset{AL}{=} \frac{1}{\sqrt{\pi}} \cdot 0 \cdot \lim_{n\to\infty} \left(\frac{2e\pi}{n}\right)^{n/2} \overset{POLICIE}{=} \underline{\underline{0}}$$

$$0 \le \left(\frac{2e\pi}{n}\right)^{n/2} \le \underbrace{\left(\frac{1}{2}\right)^{n/2}}_{\to 0}, \quad n \gg 1$$

$\boxed{\text{DŮKAZ STIRLINGOVA VZORCE}}$:

$$\lim_{s\to\infty} \frac{1}{\sqrt{s}} \left(\frac{e}{s}\right)^s \Gamma(s+1) = \lim_{s\to\infty} \int_0^\infty \frac{1}{\sqrt{s}} \exp\Big(\underbrace{s - s\log s + s\log x - x}_{= -s\left(\frac{x}{s} - 1 - \log\frac{x}{s}\right)}\Big)\,dx$$

$$\overset{\text{VOLBA}\ („a = \frac{1}{\sqrt{s}}“)}{=} \lim_{a\to 0^+} \int_0^\infty a \exp\left(-\frac{a^2x - 1 - \log(a^2x)}{a^2}\right) dx$$

$$= \left| \begin{array}{l} t = \frac{\log(a^2x)}{a} \\ dt = \frac{1}{ax}\,dx \end{array} \right| = \lim_{a\to 0^+} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{e^{at} - 1 - at}{a^2} - at\right)\right) dt$$

POKUD LZE PROHODIT INTEGRÁL A LIMITU
(COŽ OVĚŘÍME NÍŽE – VIZ (A))

PAK

$$= \int_{-\infty}^{\infty} \lim_{a\to 0^+} \ldots\ dt = \int_{-\infty}^{\infty} e^{-t^2/2}\,dt$$

[$$\lim_{a\to 0^+} \frac{e^{at} - 1 - at}{a^2} \overset{L'H(\frac{0}{0})}{=} \lim_{a\to 0^+} \frac{te^{at} - t}{2a} \overset{L'H(\frac{0}{0})}{=} \lim_{a\to 0^+} \frac{t^2 e^{at} - 0}{2} = \frac{t^2}{2}$$]

$$\overset{!}{=} \left| \begin{array}{l} x = \frac{t}{\sqrt{2}} \\ dx = \frac{1}{\sqrt{2}}\, dt \end{array} \right| = \sqrt{2} \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

$$= \sqrt{2\pi}\,!$$

↓ LAPLACEŮV INTEGRÁL

(2) JEŠTĚ ZBÝVÁ ODŮVODNIT PROHOZENÍ INTEGRÁLU A LIMITY (TJ. DLE LEBESGUEOVY VĚTY, NAJÍT INTEGROVATELNOU MAJORANTU)

POLOŽME:

$$f(t) := \begin{cases} \frac{t^2}{2} - t & \dots t \geq 0 \\ e^t - t - 1 & \dots t < 0 \end{cases}$$

$$g(t) := \frac{e^{at} - at - 1}{a^2} - at - f(t) \qquad (a \in (0,1))$$

pro $t > 0$: $g'(t) = \dfrac{a e^{at} - a}{a^2} - a - t + 1$

$$g''(t) = e^{at} - 1 \geq 0 \quad (\Rightarrow g' \text{ roste na } (0,\infty))$$

pro $t < 0$: $g'(t) = \dfrac{e^{at} - 1}{a} - a - e^t + 1$

$$g''(t) = e^{at} - e^t \geq 0 \quad (\Rightarrow g' \text{ roste na } (-\infty,0))$$

$\Rightarrow$ g je spojitá (protože f je spoj. v nule), konvexní na $(0,\infty)$ a na $(-\infty,0)$

Navíc,

$$g'_+(0) \overset{VLD}{=} \lim_{t\to 0^+} g'(t) = 1 - a > 0 \quad (a<1)$$

$$g'_-(0) \overset{VLD}{=} \lim_{t\to 0^-} g'(t) = -a < 0$$

$\Rightarrow$ g roste na $(0,\infty)$, klesá na $(-\infty,0)$, $g(0) = 0$

(obr.: ↘ 0 ↗)

$g(t) > 0$ pro $t \neq 0$

$$\Rightarrow \frac{e^{-at} - at - 1}{a^2} - at \geq f(t), \quad t \in \mathbb{R}$$

$\Rightarrow$ Máme: $\left|\exp\left(-\frac{e^{-at}-at-1}{a^2}-at\right)\right| \le \exp(-f(t))$

Zřejmě: $\exp(-f(t)) = \begin{cases} \exp\left(t-\frac{t^2}{2}\right) & \dots t \ge 0 \\ \exp(1+t-e^t) & \dots t<0 \end{cases}$

JE INTEGROVATELNÁ FCE NA $(-\infty,\infty)$

- JE SPOJITÁ NA $(-\infty,\infty)$
- „v ∞": $\int_0^\infty e^{t-t^2/2}\,dt$ JE KONVERGENTNÍ, PROTOŽE

„$e^{t-t^2/2} \ll e^{-t^2/4}$"

$$\lim_{t\to\infty} \frac{e^t e^{t^2/4}}{e^{t^2/2}} = \lim_{t\to\infty} \exp\Big(t+\overbrace{\tfrac{t^2}{4}-\tfrac{t^2}{2}}^{-\frac{t^2}{4}}\Big)$$

$$\overset{\text{„volně"}}{=} \exp\Big(\lim_{t\to\infty} \underbrace{t^2}_{\to\infty}\underbrace{\left(\tfrac{1}{t}-\tfrac{1}{4}\right)}_{\to -1/4}\Big) = 0$$

STAČÍ POUŽÍT S.K. + FAKT, ŽE

$e^{-t^2/4} \in L^1((0,\infty))$ neboť $e^{-t^2/4} \ll \frac{1}{t^2}$ „v ∞"

- „v $-\infty$":

Máme $\exp(1+t-e^t) = e\exp(t-e^t) \le e\,e^t$

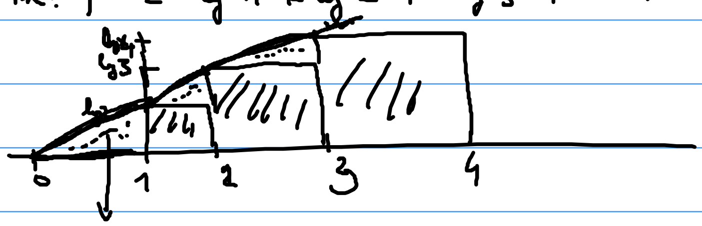
a $\int_{-\infty}^0 e^t\,dt = [e^t]_{-\infty}^0 = \underline{\underline{1}}$

$\overset{S.K.}{\Longrightarrow}$ integrál je lehký v „$-\infty$".

$\boxtimes$

IDEA PROČ TO PLATÍ:

$\log(n!) = \log 1 + \log 2 + \log 3 + \dots + \log(n)$



$\int_0^4 \log x\,dx$, ZKUSÍ SE ZDA $\lim_{n\to\infty} \int_0^n \log x\,dx - \log(n!) = 0$

... ZJISTÍ SE, ŽE NE $\Rightarrow$ NEFUNGUJE

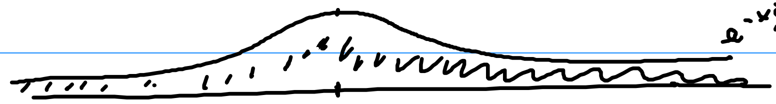
... NĚCO PODOBNÉHO ALE FUNGUJE ...

Radon–Nikodýmova věta

Př.: $f(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$, pak $\nu(E) = \int_E f(x)\,dx$

Pak • ν je míra na $\mathbb{R}$ splňující $\nu(\mathbb{R}) = 1$
(ν je PRST) viz. ! výše

obr.



TZV. NORMÁLNÍ ROZDĚLENÍ

• $\nu \ll \lambda$ ⌈ protože $\lambda(E) = 0 \Rightarrow \nu(E) = 0$
(integrál přes množinu míry nula) ⌋

Př.: Ať μ je definována jako

$$\mu(E) = \int_{E \cap [0,\infty)} e^{-x}\,dx \left(= \int_E e^{-x} \chi_{[0,\infty)}(x)\,dx \right)$$

pak • $\dfrac{d\mu}{d\lambda} = e^{-x} \chi_{[0,\infty)}(x)$

• $\mu \ll \nu$ ⌈ $\nu(E) = 0$, pak $\int_E e^{-x^2/2}\,dx = 0$, tedy

$$\lambda(E \cap [0,\infty)) \leq \int_{E \cap [0,\infty)} \underbrace{e^{-x^2/2}}_{\geq 1}\,dx = 0$$

$\Rightarrow \lambda(E \cap [0,\infty)) = 0$ (a tedy $\lambda(E) = 0$)

$\Rightarrow \mu(E) = 0$. ⌋

pozn.: obecně, platí $\tilde{\nu}(E) = \int_E f(x)\,d\lambda(x)$, ať

$f > 0$, f spojitá $\Rightarrow \lambda \ll \tilde{\nu} \ll \lambda$

• ale $\nu \not\ll \mu$

$\ulcorner$ $\mu((-1,0)) = 0$, ale $\nu((-1,0)) > 0$

• $\ulcorner$ $\int_{-1}^{0} e^{-x^2/2}\,dx \geq \int_{-1}^{0} e^{-1/2}\,dx > 0$ $\lrcorner$ $\lrcorner$

• Ba'le , $\dfrac{d\mu}{d\nu} = \sqrt{2\pi}\,\dfrac{e^{-x}}{e^{-x^2/2}}\,\chi_{[0,\infty)}(x)$

$\ulcorner$ zk:

$$\int_E \sqrt{2\pi}\,\frac{e^{-x}}{e^{-x^2/2}}\,\chi_{[0,\infty)}(x)\,d\nu \underset{3.37}{=} \int_E \cancel{\sqrt{2\pi}}\,\frac{e^{-x}}{\cancel{e^{-x^2/2}}}\,\chi_{[0,\infty)}(x)\cdot\frac{\cancel{e^{-x^2/2}}}{\cancel{\sqrt{2\pi}}}\,dx$$

$$= \mu(E) \quad \lrcorner$$