

1. Basic Topological Concepts

1.1. Topological Space, Base, Subbase

Definition. Let X be a set. We say that $\tau \subset \mathcal{P}(X)$ is a *topology* on X if

(T1) $\emptyset, X \in \tau$,

(T2) if $U, V \in \tau$, then $U \cap V \in \tau$,

(T3) if $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$.

The pair (X, τ) is then called a *topological space*. Elements of X are called *points*, elements of τ are called *open sets*. A set $V \subseteq X$ is called a *neighborhood* of a point $x \in X$ if there exists $U \in \tau$ such that $x \in U \subseteq V$. The family of all neighborhoods of the point x is denoted by $\mathcal{U}(x)$. A family of sets $\mathcal{B} \subseteq \tau$ is called a *base* of the space X if every $U \in \tau$ can be expressed as a union of some $\mathcal{U} \subseteq \mathcal{B}$. A family $\mathcal{S} \subseteq \tau$ is called a *subbase* if the system

$$\left\{ \bigcap \mathcal{F} : \mathcal{F} \subseteq \mathcal{S}, \mathcal{F} \text{ is finite} \right\}$$

is a base.

Examples. • If (X, ρ) is a metric space, then the family of all ρ -open sets forms a topology on X . We then say that this *topology is generated by the metric ρ* . A base of the topology generated by a metric is for example $\{B(x, r) : x \in X, r > 0\}$. A topological space is called *metrizable* if its topology is generated by some metric.

- On \mathbb{R} with the usual metric, a base of the topology consists of open intervals; a subbase is for example the system $\{(-\infty, b), (a, \infty) : a, b \in \mathbb{R}\}$. On \mathbb{R}^2 with the usual metric, a base of the topology is for example $\{(a, b) \times (c, d) : a < b, c < d\}$; a subbase of the topology of the plane is for example $\{\mathbb{R} \times (a, b), (a, b) \times \mathbb{R} : a < b\}$.
- On every set X there are two trivial topologies. The pair $(X, \mathcal{P}(X))$ is called a *discrete space* ($\mathcal{P}(X)$ is the discrete topology). The pair $(X, \{\emptyset, X\})$ is called an *indiscrete space* ($\{\emptyset, X\}$ is the indiscrete topology). Every discrete space is metrizable. An indiscrete space with at least two points is not metrizable.

Proposition 1 (Properties of a base). *If (X, τ) is a topological space and \mathcal{B} is its base, then*

(B1) *for every $U, V \in \mathcal{B}$ and $x \in U \cap V$ there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$,*

(B2) $\bigcup \mathcal{B} = X$.

If X is a set and $\mathcal{B} \subseteq \mathcal{P}(X)$ is a family of sets satisfying conditions (B1) and (B2), then there exists a unique topology on X whose base is \mathcal{B} .

Důkaz. The proof was presented, it will be examined. □

Corollary 2. *If X is a set and $\mathcal{S} \subset \mathcal{P}(X)$ satisfies $\bigcup \mathcal{S} = X$, then there exists exactly one topology on X whose subbase is \mathcal{S} .*

Důkaz. The proof was presented, it will be examined. □

Examples. • Let $(X, <)$ be a linearly ordered set. Then the *topology generated by the order $<$* is the topology τ generated by the base

$$\mathcal{B} = \{(a, b) : a, b \in X, a < b\} \cup \{(\leftarrow, b) : b \in X\} \cup \{(a, \rightarrow) : a \in X\} \cup \{X\}.$$

(Elements of \mathcal{B} are called open intervals.) We then say that (X, τ) is an *ordered topological space*.

- The *Sorgenfrey line* is the set \mathbb{R} with the topology generated by the base $\mathcal{B} = \{[a, b) : a < b\}$.

Proposition 3 (Characterization of open sets via neighborhoods). *Let (X, τ) be a topological space and let $U \subseteq X$. Then $U \in \tau$ if and only if*

$$\forall x \in U \exists V \in \mathcal{U}(x) : V \subseteq U.$$

Důkaz. The proof was presented, it will be examined. □

Proposition 4 (Properties of the system of all neighborhoods). *If (X, τ) is a topological space, then for every $x \in X$ the family $\mathcal{U}(x)$ satisfies:*

(U1) $\mathcal{U}(x) \neq \emptyset$ and $x \in \bigcap \mathcal{U}(x)$,

(U2) if $U \in \mathcal{U}(x)$ and $U \subseteq V \subseteq X$, then $V \in \mathcal{U}(x)$,

(U3) if $U, V \in \mathcal{U}(x)$, then $U \cap V \in \mathcal{U}(x)$,

(U4) for every $U \in \mathcal{U}(x)$ there exists $V \in \mathcal{U}(x)$ such that for every $y \in V$ we have $U \in \mathcal{U}(y)$.

If X is an arbitrary set and families $\mathcal{U}(x) \subseteq \mathcal{P}(X)$, $x \in X$, satisfy conditions (U1)–(U4), then there exists a unique topology τ on X whose neighborhood systems are $\{\mathcal{U}(x)\}_{x \in X}$.

Důkaz. The proof was presented, it will be examined. □

Definition. Let (X, τ) be a topological space. A family of sets $\mathcal{B}(x)$ is called a *neighborhood base* at the point x if $\mathcal{B}(x) \subseteq \mathcal{U}(x)$ and for every $V \in \mathcal{U}(x)$ there exists $U \in \mathcal{B}(x)$ such that $U \subseteq V$. An indexed family $\{\mathcal{B}(x)\}_{x \in X}$ is called a *neighborhood base of the space X* .

Proposition 5 (Properties of a neighborhood base). *If (X, τ) is a topological space and $\{\mathcal{B}(x)\}_{x \in X}$ is a neighborhood base, then for every $x \in X$ the family $\mathcal{B}(x)$ satisfies*

(O1) $\mathcal{B}(x) \neq \emptyset$ and $x \in \bigcap \mathcal{B}(x)$,

(O2) for every $U, V \in \mathcal{B}(x)$ there exists $W \in \mathcal{B}(x)$ such that $W \subseteq U \cap V$,

(O3) for every $U \in \mathcal{B}(x)$ there exists $V \in \mathcal{B}(x)$ such that for every $y \in V$ there exists $W \in \mathcal{B}(y)$ with $W \subseteq U$.

If X is an arbitrary set and families $\mathcal{B}(x) \subseteq \mathcal{P}(X)$, $x \in X$, satisfy conditions (O1), (O2), and (O3), then there exists a unique topology τ on X with neighborhood base $\{\mathcal{B}(x)\}_{x \in X}$.

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

Definition. The *weight* of a topological space X , denoted by $w(X)$, is the smallest cardinality of a base. The more precise notation $w(X, \tau)$ is used when the topology τ is not clear from the context. The *character* at a point $x \in X$, denoted by $\chi(x, X)$, is the smallest cardinality of a neighborhood base at the point x . The character of the space X is the supremum of the characters at its points.

Examples. • If X is metrizable, then it has countable character. Moreover, $w(X) \leq \omega$ if and only if X is separable.

- If X is discrete, then $\chi(X) = 1$ and $w(X) = |X|$.
- If X is indiscrete, then $w(X) = \chi(X) = 1$.

the end of 1. lecture (29. 9. 2025)

Proposition 6. *Let X be a topological space. Then $\chi(X) \leq w(X) \leq 2^{|X|}$.*

Důkaz. The proof was presented, it will be examined. □

1.2. Interior, Closure and Boundary, Dense and Nowhere Dense Sets, Accumulation and Isolated Points, Continuous Mappings

Definition. A set F in a topological space (X, τ) is called *closed* if $X \setminus F$ is open. A set is called *clopen* if it is both open and closed. If $A \subseteq X$, then the *closure* of the set A is the set

$$\bar{A} := \bigcap \{F : F \text{ is closed, } A \subseteq F\}.$$

The *interior* of the set A is the set

$$\text{Int } A = \bigcup \{U \in \tau : U \subseteq A\}.$$

The *boundary* of the set A is $\partial A = \bar{A} \cap \overline{X \setminus A}$.

Remark. It is easy to see that \emptyset and X are clopen sets and that the family of closed sets is closed under finite unions and arbitrary intersections. The closure and the boundary are closed sets, while the interior is an open set. A set is closed if and only if it equals its closure, and it is open if and only if it equals its interior.

Proposition 7 (Relation between interior and closure). *Let X be a topological space. Then*

$$X \setminus \bar{A} = \text{Int}(X \setminus A) \quad \text{and} \quad X \setminus \text{Int } A = \overline{X \setminus A}.$$

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. \square

Proposition 8 (Characterization of closure). *Let X be a topological space, $x \in X$, $A \subseteq X$, and let $\mathcal{B}(x)$ be a neighborhood base at the point x . Then the following conditions are equivalent:*

- (1) $x \in \bar{A}$,
- (2) for every $U \in \mathcal{U}(x)$ we have $U \cap A \neq \emptyset$,
- (3) for every $U \in \mathcal{B}(x)$ we have $U \cap A \neq \emptyset$.

In particular, if $U \subset X$ is open, then

$$U \cap A = \emptyset \Leftrightarrow U \cap \bar{A} = \emptyset.$$

Důkaz. The proof was presented, it will be examined. \square

Proposition 9 (Characterization of boundary). *Let X be a topological space, $A \subseteq X$, and $x \in X$. Then*

$$x \in \partial A \Leftrightarrow \forall U \in \mathcal{U}(x) : (U \cap A \neq \emptyset \ \& \ U \cap (X \setminus A) \neq \emptyset).$$

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. \square

Definition. Let X be a topological space and $A \subset X$. The set A is called *dense* if $\bar{A} = X$. The set A is called *nowhere dense* if $X \setminus \bar{A}$ is dense in X . The *density* of the space X is the smallest cardinality of a dense subset; it is denoted by $d(X)$. We say that X is *separable* if $d(X) \leq \omega$.

Proposition 10 (Characterization of dense and nowhere dense sets). *Let X be a topological space and $A \subset X$.*

- A is dense in X if and only if every nonempty open set $U \subseteq X$ intersects A ,
- A is nowhere dense in X if and only if $\text{Int}(\bar{A}) = \emptyset$,
- A is nowhere dense in X if and only if every nonempty open set V contains a nonempty open set U that is disjoint from A .

Důkaz. The proof was presented, it will be examined. \square

Proposition 11 (Relation between weight and density). *For every topological space X we have $d(X) \leq w(X)$. If X is metrizable, then $d(X) = w(X)$.*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. \square

Definition. A point $x \in A \subseteq X$ is called an *isolated point* of the set A if there exists an open set $U \subseteq X$ such that $U \cap A = \{x\}$. A point $x \in X$ is called an *accumulation point* of the set A if every neighborhood U of the point x intersects $A \setminus \{x\}$. The set of all accumulation points of the set A is denoted by A' (the so-called derivative of the set A).

Examples. In a discrete space all points are isolated and there are no accumulation points. If $X = \mathbb{R}$ and $A = \mathbb{Q}$, then $A' = \mathbb{R}$ and no point of A is isolated.

Proposition 12 (Properties of the derivative). *Let X be a topological space and $A \subset X$. Then*

$$\overline{A} = A \cup A' \quad \text{and} \quad (A \cup B)' = A' \cup B'.$$

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. \square

Definition. Let (X, τ) and (Y, σ) be two topological spaces and let $f: X \rightarrow Y$ be a function. The function f is called *continuous* if for every $U \in \sigma$ we have $f^{-1}(U) \in \tau$. The mapping f is called a *homeomorphism* if f is a bijection and both f and f^{-1} are continuous. The mapping f is called an *embedding* if its restriction $f: X \rightarrow f(X)$ is a homeomorphism onto the subspace $f(X) \subset Y$. We say that f is *continuous at the point x* if for every neighborhood V of the point $f(x)$ there exists a neighborhood U of the point x such that $f(U) \subseteq V$. The mapping f is called *open/closed* if the set $f(M)$ is open/closed in Y for every open/closed set $M \subseteq X$.

Remark. If X, Y, Z are topological spaces and $x \in Z$, then if a function $f: X \rightarrow Y$ is continuous (at the point x), its restriction $f|_Z$ is also continuous (at the point x).

Proposition 13 (Characterization of continuous mappings). *Let (X, τ) and (Y, σ) be topological spaces and let $f: X \rightarrow Y$ be a mapping. Let \mathcal{B} be a base of Y and \mathcal{S} a subbase of Y . Then the following conditions are equivalent:*

- (1) f is continuous,
- (2) preimages of sets from the subbase \mathcal{S} are open,
- (3) f is continuous at every point,
- (4) preimages of sets from the base \mathcal{B} are open,
- (5) preimages of closed sets are closed,
- (6) for every $A \subseteq X$ we have $f(\overline{A}) \subseteq \overline{f(A)}$.

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. \square

Remark. The composition of continuous mappings is continuous.

the end of 2. lecture (6. 10. 2025)

Definition. If (X, τ) is a topological space and $Y \subset X$, let $\sigma := \{U \cap Y : U \in \tau\}$. Then σ is a topology on Y , and we say that (Y, σ) is a *subspace* of X .

Examples. For metrizable spaces it holds that the weight equals the density and that a subspace of a separable space is separable. For non-metrizable spaces these statements need not hold. The following examples will be discussed gradually in exercises.

- If X is the double arrow space or the Sorgenfrey line, then $d(X) = \omega < 2^\omega = w(X)$. Moreover, every subspace of X is separable.
- If $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ is an uncountable family of infinite sets that is almost disjoint, then for the corresponding Mrówka–Isbell space $X := \psi(\mathcal{A})$ we have $d(X) = \omega < |\mathcal{A}| = w(X)$. Moreover, X is separable but contains a non-separable subspace (more precisely, $\mathcal{A} \subset \psi(\mathcal{A})$ is an uncountable closed subspace that is discrete).

1.3. Nets and Convergence

Definition. A pair (I, \leq) is called an *upward directed* set if \leq is a partial order on I (i.e. a binary reflexive, transitive, and weakly antisymmetric relation) with the property that for $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. By the symmetric symbol \geq we naturally mean the inverse relation. A *net* in a topological space X is a mapping from some nonempty directed set into X . A special case is a *sequence* (which is a net indexed by the natural numbers). We say that a net $(x_i)_{i \in I}$ *converges* to a point $x \in X$ if for every $U \in \mathcal{U}(x)$ there exists $i \in I$ such that for every $j \geq i$ we have $x_j \in U$. The point x is then called the *limit of the net* $(x_i)_{i \in I}$.

Proposition 14 (Characterization of closure via convergence of nets). *Let X be a topological space and $A \subseteq X$. Then*

- $x \in \bar{A}$ if and only if x is the limit of a net consisting of points from A ,
- A is closed if and only if every convergent net consisting of points in A has its limit in A .

Důkaz. The proof was presented, it will be examined. □

Proposition 15 (Characterization of continuity via convergence of nets). *Let X and Y be topological spaces. A mapping $f: X \rightarrow Y$ is continuous at the point x if and only if for every net $(x_i)_{i \in I}$ converging to x the net $(f(x_i))_{i \in I}$ converges to $f(x)$.*

Důkaz. The proof was presented, it will be examined. □

Example. Statements 14 and 15 do not hold if we replace the word “net” by the word “sequence”. Indeed, let X be an uncountable set with the topology

$$\tau := \{\emptyset\} \cup \{A \subset X : X \setminus A \text{ is countable}\}.$$

This topology τ is called the *co-countable topology*. Then for any uncountable set $A \subsetneq X$ we have that A is not closed, but at the same time every convergent sequence consisting of points from A is eventually constant. Furthermore, if we consider the discrete topology on $\{0, 1\}$ and define a function $f: X \rightarrow \{0, 1\}$ by $f(x) = 1$ for $x \in A$ and $f(x) = 0$, then f is not continuous, but whenever $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in A , we have

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right).$$

Důkaz. The proof was presented, it will be examined. □

1.4. Separation Axioms

Definition. A topological space X is called

- T_0 if for every two distinct points $x, y \in X$ there exists an open set U such that $|U \cap \{x, y\}| = 1$,
- T_1 if for every two distinct points $x, y \in X$ there exists an open set U such that $x \in U$ and $y \notin U$,
- T_2 (or *Hausdorff*) if for every two distinct points $x, y \in X$ there exist disjoint open sets U, V such that $x \in U$ and $y \in V$,
- *regular* if for every closed set $F \subseteq X$ and every point $x \in X \setminus F$ there exist disjoint open sets U, V such that $x \in U$ and $F \subseteq V$,
- *normal* if for every two disjoint closed sets E, F there exist disjoint open sets U, V such that $E \subseteq U$ and $F \subseteq V$,
- *completely regular* if for every closed set F and every point $x \in X \setminus F$ there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(F) \subseteq \{1\}$,

- T_3 if it is regular and T_1 ,
- $T_{3\frac{1}{2}}$ (or *Tychonoff*) if it is completely regular and T_1 ,
- T_4 if it is normal and T_1 .

Remark. The following implications hold:

$$T_4 \implies T_{3\frac{1}{2}} \implies T_3 \implies T_2 \implies T_1 \implies T_0,$$

however the first implication is relatively difficult to prove (see Corollary 19), while the others are easy. None of the listed implications can be reversed; this is more difficult to show in the first three cases (see Examples 1.4 and the example of the Niemytzki plane, which will be presented in exercises; the example showing that the second implication cannot be reversed is the most difficult and will be omitted here, it is written in the lecture notes).

It is easy to see that the properties $T_{3\frac{1}{2}}$, T_3 , T_2 , T_1 , and T_0 are preserved under subspaces, and that the property T_4 is preserved under closed subspaces. In general, normality is not preserved under subspaces; a corresponding counterexample will be mentioned later.

Lemma 16. *Let (X, τ) be a topological space. Then the following hold:*

- X is $T_1 \iff$ every singleton subset of X is closed \iff every finite subset of X is closed.
- X is $T_2 \iff \forall x, y \in X, x \neq y \exists U \in \mathcal{U}(x): x \in U$ and $y \notin \bar{U}$.
- X is regular $\iff \forall x \in X \forall U \in \mathcal{U}(x) \exists V \in \mathcal{U}(x): \bar{V} \subseteq U$.
- X is normal $\iff \forall V \in \tau \forall E \subseteq V$ closed $\exists U \in \tau: E \subseteq U \subseteq \bar{U} \subseteq V$.

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. \square

Proposition 17 (Uniqueness of the limit of a net). *A topological space is T_2 if and only if every net has at most one limit.*

Důkaz. The proof was presented, it will be examined. \square

Věta 18 (Urysohn's Lemma). *A topological space X is normal if and only if for every two disjoint closed sets E and F there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f(E) \subseteq \{0\}$ and $f(F) \subseteq \{1\}$.*

Důkaz. The proof was presented, it will be examined. \square

Corollary 19. *If a topological space is T_4 , then it is $T_{3\frac{1}{2}}$.*

Důkaz. The proof was presented, it will be examined. \square

the end of 3. lecture (14. 10. 2025)

Definition. We say that a topological space is *zero-dimensional* if it has a neighborhood base consisting of clopen sets.

Examples. • The Sierpiński space (i.e. the space $X = \{0, 1\}$ with topology $\tau = \{\emptyset, X, \{0\}\}$) is T_0 but not T_1 .

- An example of a topological space that is T_1 but not T_2 is an infinite set X with the *cofinite topology*, i.e. with topology $\tau := \{\emptyset\} \cup \{A \subset X: X \setminus A \text{ is finite}\}$.
- An example of a topological space that is T_2 but not T_3 is the set $S := (0, 1)^2 \cup \{(0, 0)\}$ with topology generated by a neighborhood base defined as follows:
 - for $x \in (0, 1)^2$ we set $\mathcal{B}(x) := \{B_\rho(x, \varepsilon): \rho \text{ is the Euclidean metric and } \varepsilon > 0\}$,
 - for $x = (0, 0)$ we set $\mathcal{B}(x) := \{[0, 1/2) \times [0, 1/n) \cap S: n \in \mathbb{N}\}$.

- Every zero-dimensional T_1 topological space is $T_{3\frac{1}{2}}$.

Důkaz. The proof was presented, it will be examined.

□

Proposition 20 (Metrisable spaces are normal). *Metrisable spaces are T_4 .*

Důkaz. The proof was presented, it will be examined.

□

2. Operations with Topological Spaces and Mappings

2.1. Projectively and Inductively Generated Topologies

Definition. Let X be a set and let τ, σ be two topologies on X . We say that τ is *larger* (*finer*, *stronger*) than σ if $\tau \supseteq \sigma$. In this case the topology σ is called *smaller* (*coarser*, *weaker*).

The discrete topology is the largest of all topologies, the indiscrete topology is the smallest.

Definition (Projective and inductive generation). Let X be a set, (X_i, τ_i) , $i \in I$, a family of topological spaces, and let $f_i: X \rightarrow X_i$, $i \in I$, be a family of mappings. A topology τ on the set X is called *projectively generated* by the mappings f_i if τ is the smallest topology such that all mappings $f_i: (X, \tau) \rightarrow (X_i, \tau_i)$ are continuous. If $f_i: X_i \rightarrow X$, then a topology τ on X is called *inductively generated* if τ is the largest topology such that all mappings $f_i: (X_i, \tau_i) \rightarrow (X, \tau)$ are continuous.

Věta 21 (Existence and properties of a projectively generated topology). *Let X be a set, (X_i, τ_i) , $i \in I$, a family of topological spaces, and let $f_i: X \rightarrow X_i$, $i \in I$, be a family of mappings. Then there exists exactly one topology τ projectively generated by the family of mappings f_i , $i \in I$. This topology has the following properties:*

- (P1) A subbase of the topology τ is $\{f_i^{-1}(U) : U \in \tau_i\}$,
- (P2) Whenever (Y, σ) is a topological space and $g: Y \rightarrow X$ is given, then the mapping g is continuous if and only if for every $i \in I$ the mapping $f_i \circ g$ is continuous,
- (P3) Whenever a net $(x_\alpha)_{\alpha \in A}$ in X and a point $x \in X$ are given, then $x_\alpha \rightarrow x$ in the space (X, τ) if and only if for every $i \in I$ we have $f_i(x_\alpha) \rightarrow f_i(x)$.

Důkaz. The proof was presented, it will be examined. □

Examples. • If Y is a subspace of a topological space X , then the topology on Y is projectively generated from the topology on X via the identity mapping $i: Y \rightarrow X$.

- A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is an embedding if and only if it is injective and the topology τ is projectively generated by the mapping f .
- Let X be a topological space. We denote by $C_p(X)$ the topological space $(C(X), \tau_p)$, where $C(X) := \{f: X \rightarrow \mathbb{R} : f \text{ is continuous}\}$ and τ_p is the topology projectively generated by the mappings $\delta_x: C(X) \rightarrow \mathbb{R}$, $x \in X$, defined by $\delta_x(f) := f(x)$. Then for every net of functions $(f_i)_{i \in I}$ in $C_p(X)$ and every $f \in C_p(X)$ we have $f_i \rightarrow f$ if and only if $f_i(x) \rightarrow f(x)$ for every $x \in X$. We say that τ_p is the *topology of pointwise convergence*.

Věta 22 (Existence and properties of an inductively generated topology). *Let X be a set, (X_i, τ_i) , $i \in I$, a family of topological spaces, and let $f_i: X_i \rightarrow X$, $i \in I$, be a family of mappings. Then there exists exactly one topology τ inductively generated by the family of mappings f_i , $i \in I$. This topology has the following properties:*

- (I1) $\tau = \{M \subset X : f_i^{-1}(M) \in \tau_i, i \in I\}$,
- (I2) Whenever (Y, σ) is a topological space and a mapping $g: X \rightarrow Y$ is given, then g is continuous if and only if for every $i \in I$ the mapping $g \circ f_i$ is continuous.

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

2.2. Product

Definition. If (X_i, τ_i) are topological spaces, then their product is the topological space (X, τ) , where $X = \prod X_i$ and the topology τ is projectively generated by the projections $\pi_i: X \rightarrow X_i$. It is denoted by $\prod_{i \in I} X_i$. A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is then called an *embedding* if f is injective and the topology τ is projectively generated by the mapping f .

Remark. A base \mathcal{B} of the product has the following form:

$$\mathcal{B} = \{\pi_{i_1}^{-1}(U_1) \cap \cdots \cap \pi_{i_n}^{-1}(U_n) : i_1, \dots, i_n \in I, U_1 \in \tau_{i_1}, \dots, U_n \in \tau_{i_n}, n \in \mathbb{N}\},$$

and the intersection of preimages of projections can be rewritten as

$$\pi_{i_1}^{-1}(U_1) \cap \cdots \cap \pi_{i_n}^{-1}(U_n) = \left\{ x \in \prod X_i : x(i_1) \in U_1, \dots, x(i_n) \in U_n \right\}.$$

Convergence of nets in the product topology is straightforward: a net $(x_j)_{j \in J}$ converges to x if and only if for every $i \in I$ the net $(x_j(i))_{j \in J}$ converges to $x(i)$. Thus the product topology corresponds to pointwise convergence. For this reason, the product topology is sometimes called the topology of pointwise convergence. The closure of a set of the form $\prod A_i$ is $\prod \bar{A}_i$.

the end of 4. lecture (20. 10. 2025)

Definition. Given a family of mappings $f_i : X_i \rightarrow Y_i$, $i \in I$, we define the *product mapping* $\prod_I f_i : \prod_I X_i \rightarrow \prod_I Y_i$ by

$$\prod_I f_i(\{x_i\}_{i \in I}) = \{f_i(x_i)\}_{i \in I}.$$

Given a family of mappings $f_i : X \rightarrow Y_i$, $i \in I$, we define the *diagonal mapping* $\Delta_I f_i : X \rightarrow \prod_I Y_i$ by

$$\Delta_I f_i(x) = \{f_i(x)\}_{i \in I}.$$

Proposition 23. Let $Z, Y_i, i \in I$, and $X_i, i \in I$, be topological spaces.

- (i) A mapping $f: Z \rightarrow \prod_I X_i$ is continuous if and only if all compositions of f with the projections $\pi_i : \prod_I X_i \rightarrow X_i$ are continuous.
- (ii) If $f_i: Y_i \rightarrow X_i$, $i \in I$, are continuous mappings, then their product $\prod_I f_i: \prod_I Y_i \rightarrow \prod_I X_i$ is continuous.
- (iii) If $f_i: Z \rightarrow X_i$, $i \in I$, are continuous mappings, then their diagonal product $\Delta_I f_i: Z \rightarrow \prod_I X_i$ is continuous.

Důkaz. The proof was presented, it will be examined. □

Corollary 24. If X is a topological space and $f, g: X \rightarrow \mathbb{R}$ are continuous mappings, then $f + g$, $f - g$, $f \cdot g$, $\max\{f, g\}$, $\min\{f, g\}$, and $|f|$ are continuous. If g is nonzero, then f/g is also continuous.

Důkaz. The proof was presented, it will be examined. □

Proposition 25 (Characterization of Hausdorff spaces). A topological space X is Hausdorff if and only if $\{(x, x) : x \in X\}$ is closed in $X \times X$.

Důkaz. The proof was presented, it will be examined. □

2.3. Sum and Quotient

Definition. The sum of spaces X_i , $i \in I$, is the space $\bigsqcup_I X_i$ consisting of points (i, x) , where $i \in I$ and $x \in X_i$. If all sets X_i are disjoint, one usually takes the points x instead of the pairs (i, x) .

If (X_i, τ_i) , $i \in I$, are topological spaces, then their *topological sum* is the topological space $(\bigsqcup_I X_i, \tau)$, where the topology τ is defined by

$$\tau := \left\{ \bigsqcup_I O_i : O_i \subset X_i \text{ are open} \right\}.$$

The topological sum is denoted by $\bigoplus_I X_i$.

(Warning: there is no uniform agreement on the notation for the topological sum; different symbols are used in the literature. The notation used above is somewhat inconvenient, since for example in Banach space theory the symbol $\bigoplus_I X_i$ denotes a Banach space which, as a topological space, is the product of the X_i rather than their topological sum.)

Remark. The topology on $X := \bigoplus_I X_i$ is inductively generated by the canonical embeddings $X_i \ni x \mapsto (i, x) \in \bigsqcup_I X_i$.

If the X_i are disjoint, then a set U in the topological sum $\bigoplus_{i \in I} X_i$ is open if and only if for every $i \in I$ the set $U \cap X_i$ is open in X_i . The sets X_i are clopen in X . If $f_i : X_i \rightarrow Y_i$, $i \in I$, are continuous mappings, then $\bigoplus_I f_i : \bigoplus_I X_i \rightarrow \bigoplus_I Y_i$ is continuous.

Definition. If (X, τ) is a topological space and $E \subseteq X \times X$ is an equivalence relation, then the *quotient topology* on X/E is inductively generated by the mapping $x \mapsto [x]_E$.

A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called a *quotient mapping* if f is surjective and the topology σ is inductively generated by the mapping f .

Remark. A more intuitive description of the quotient topology is again direct. A set U in the quotient space Y is open if and only if $f^{-1}(U)$ is open in X .

If $f : X \rightarrow Y$ is a quotient mapping, then one can naturally consider an equivalence relation \sim on X defined by $x \sim y$ if and only if $f(x) = f(y)$. Then X/\sim is naturally homeomorphic to Y .

Examples. • If we define an equivalence relation on \mathbb{R} by xEy if and only if $\{x, y\} \in \mathbb{Q}$ or $\{x, y\} \subset \mathbb{R} \setminus \mathbb{Q}$, then \mathbb{R}/E is a two-point indiscrete space. In particular, a quotient of a T_4 space need not even be T_0 .

- If we define an equivalence relation on \mathbb{R} by xEy if and only if $x - y \in \mathbb{Z}$, then the quotient space \mathbb{R}/E is homeomorphic to the circle.

Proposition 26 (Characterization of a quotient mapping). *Let (X, τ) and (Y, σ) be topological spaces and let $f : X \rightarrow Y$ be a mapping. Then f is a quotient mapping if and only if f is surjective and for every $V \subseteq Y$ we have $V \in \sigma \iff f^{-1}(V) \in \tau$.*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

Proposition 27 (Sufficient condition for a quotient mapping). *If a mapping $f : X \rightarrow Y$ is continuous, open (or closed), and surjective, then it is a quotient mapping.*

Důkaz. The proof was presented, it will be examined. □

Remark. The open mapping theorem states that a continuous surjective linear mapping between Banach spaces is already open. A quotient mapping between topological spaces need not be either open or closed.

Remark. A continuous image (and hence also a quotient) of a separable space is separable.

Overview of preservation of properties by individual operations. The symbol (+) means preservation at least under countable products or sums.

	$T_0, T_1, T_2, T_3, T_{3\frac{1}{2}}$	T_4	separable	countable base	countable character	metrizable
subspace	+	-	-	+	+	+
(countable) sum	+	+	-(+)	-(+)	+	+
quotient	-	-	+	-	-	-
(countable) product	+	-	-(+)	-(+)	-(+)	-(+)

(Some material was/will be covered in lectures, some was/will be covered in exercises.)

2.4. Extension of Continuous Functions

Proposition 28. *Let X, Y be topological spaces and let $f, g: X \rightarrow Y$ be continuous mappings. If Y is Hausdorff, then the set $\{x \in X: f(x) = g(x)\}$ is closed.*

Důkaz. The proof was presented, it will be examined. □

Corollary 29 (Uniqueness of continuous extension). *Thus, if $f: X \rightarrow Y$ is continuous, Y is Hausdorff, and $S \subseteq X$ is dense, then the function $f \upharpoonright S$ has a unique continuous extension to the whole space X .*

Důkaz. The proof was presented, it will be examined. □

Proposition 30. *If X is a topological space, $f_n: X \rightarrow \mathbb{R}$ are continuous, and the sequence (f_n) converges uniformly to f , then f is continuous.*

Důkaz. The proof was presented, it will be examined. □

the end of 5. lecture (27. 10. 2025)

Věta 31 (Tietze–Urysohn). *If X is a normal space and $F \subseteq X$ is closed, then every continuous function $f: F \rightarrow \mathbb{R}$ can be continuously extended to the whole space X , i.e. there exists a continuous function $\tilde{f}: X \rightarrow \mathbb{R}$ such that $\tilde{f} \upharpoonright F = f$.*

Důkaz. The proof was presented, it will be examined. □

3. Compactness

3.1. Compact Spaces

Definition. A system of sets \mathcal{S} is called a *cover* of a space X if $\bigcup \mathcal{S} = X$. Any subsystem \mathcal{S} that is a cover is called a *subcover*. A cover is called *open* if its elements are open sets. A topological space is called *compact* if from every open cover one can select a finite subcover. A topological space is called *countably compact* if from every countable open cover one can select a finite subcover. A topological space is called *Lindelöf* if from every open cover one can select a countable subcover.

We say that a system $\mathcal{F} \subseteq \mathcal{P}(X)$ is *centered* if $F_1 \cap \dots \cap F_n \neq \emptyset$ for $n \in \mathbb{N}$ and $F_1, \dots, F_n \in \mathcal{F}$.

Definition. Let X be a topological space, (I, \leq) an upward directed set, and $(x_i)_{i \in I}$ a net. A point $x \in X$ is called an *accumulation point of the net* $(x_i)_{i \in I}$ if for every $U \in \mathcal{U}(x)$ and every $i \in I$ there exists $j \geq i$ such that $x_j \in U$.

Let (J, \leq) be another upward directed set. We say that a mapping $\varphi : J \rightarrow I$ is *cofinal* if for every $i_0 \in I$ there exists $j_0 \in J$ such that $\varphi([j_0, \rightarrow]) \subset [i_0, \rightarrow]$ (i.e. for every $j \in J$ with $j_0 \leq j$ we have $i_0 \leq \varphi(j)$). If a cofinal mapping $\varphi : J \rightarrow I$ is given, then we say that $(x_{\varphi(j)})_{j \in J}$ is a *subnet* of the net $(x_i)_{i \in I}$.

Remark. If a sequence $(x_n)_{n=1}^\infty$ in a space X is given, then subsequences are a special case of subnets. However, there also exist subnets that are not subsequences. For example, $(x_{F(t)})_{t \in \mathbb{R}}$, where $F : \mathbb{R} \rightarrow \mathbb{N}$ is any mapping satisfying $\lim_{t \rightarrow \infty} F(t) = \infty$.

It is easy to see that if a net (x_i) converges to a point x , then every subnet also converges to the point x .

Věta 32 (Characterization of compactness). *For a topological space X , the following conditions are equivalent.*

- (a) X is compact.
- (b) Every centered system of closed sets has a nonempty intersection.
- (c) Every net in X has a convergent subnet.

Důkaz. The proof was presented, it will be examined. □

Lemma 33. *Let X be a topological space and $x \in X$. Then x is an accumulation point of a net $(x_i)_{i \in I}$ if and only if there exists a subnet of $(x_i)_{i \in I}$ that converges to the point x .*

Důkaz. The proof was presented, it will be examined. □

Věta 34 (Characterization of countable compactness). *For a topological space X , the following conditions are equivalent.*

- (a) X is countably compact.
- (b) Every countable centered system of closed sets has a nonempty intersection.
- (c) Every sequence in X has a convergent subnet.

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

Věta 35 (Characterization of the Lindelöf property). *For a topological space X , the following conditions are equivalent.*

- (a) X is Lindelöf.

(b) Every countably centered system of closed sets has a nonempty intersection.

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. \square

Proposition 36 (Preservation of properties). *Compactness, countable compactness, and the Lindelöf property are inherited by closed subspaces and continuous images.*

Důkaz. The proof was presented, it will be examined. \square

Corollary 37 (Attainment of extrema). *A continuous real-valued function on a nonempty (countably) compact space attains its maximum and minimum.*

Důkaz. The proof was presented, it will be examined. \square

Examples. • If (X, \leq) is linearly ordered and every (even empty) subset $A \subset X$ has a supremum and an infimum, then X with the topology generated by the order is a compact space.

Důkaz. The proof was presented, it will be examined. \square

- $[0, \alpha]$ is a compact space for every ordinal α .

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. \square

the end of 6. lecture (3. 11. 2025)

Lemma 38 (Alexander). *Let X be a topological space and \mathcal{S} its subbase. Suppose that from every cover $\mathcal{U} \subseteq \mathcal{S}$ one can select a finite subcover. Then X is compact.*

Důkaz. The proof was presented, it will be examined. \square

Věta 39 (Tychonoff). *The product of compact topological spaces is compact.*

Důkaz. The proof was presented, it will be examined. \square

Remark. In ZF, Tychonoff's theorem is equivalent to the Axiom of Choice. Tychonoff's theorem restricted only to Hausdorff spaces is, in ZF, equivalent to the Boolean prime ideal theorem.

In the following statements we need compact spaces to be Hausdorff.

Proposition 40 (Compact sets are closed in Hausdorff spaces). *If X is a Hausdorff topological space and $K \subseteq X$ is compact, then K is closed in X .*

Důkaz. The proof was presented, it will be examined. \square

Proposition 41 (Automatic homeomorphism). *Let X, Y be Hausdorff compact topological spaces and let $f: X \rightarrow Y$ be continuous. If f is surjective, then f is a quotient mapping. If f is bijective, then f is a homeomorphism.*

Důkaz. The proof was presented, it will be examined. \square

Remark. Thus a compact Hausdorff topology on a given set is minimal among Hausdorff topologies and maximal among compact ones.

Věta 42 (Sufficient condition for normality). *A regular Lindelöf topological space is normal. A Hausdorff compact space is T_4 .*

Důkaz. The proof was presented, it will be examined. \square

Proposition 43 (A continuous image of a compact space does not increase weight). *Let X, Y be Hausdorff topological spaces, X compact, and $f: X \rightarrow Y$ a continuous surjective mapping. Then $w(Y) \leq w(X)$.*

3.2. Spaces of Continuous Functions on a Compact Space

For two topological spaces X, Y we denote by $C(X, Y)$ the set of all continuous functions from X to Y . In the case $Y = \mathbb{R}$, we write simply $C(X)$. For a compact topological space K , the space $C(K)$ together with the supremum norm forms a Banach space whose properties are closely related to the topological properties of the compact space K .

Proposition 44 (Dini's criterion for uniform convergence). *Let K be a compact topological space, and let $f_n: K \rightarrow \mathbb{R}$ be a sequence of continuous functions such that $f_{n+1} \geq f_n$ and that converge pointwise to a continuous function $f: K \rightarrow \mathbb{R}$. Then (f_n) converges uniformly to f .*

Důkaz. The proof was presented, it will be examined. □

Lemma 45 (On the square root). *There exists a sequence of polynomials that converges uniformly to the function \sqrt{t} on $[0, 1]$.*

Důkaz. The proof was presented, it will be examined. □

the end of 7. lecture (10. 11. 2025)

Definition. We say that a system of functions \mathcal{F} mapping X into Y *separates points* if for any two distinct points $x, y \in X$ there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.

A ring and a lattice are algebraic notions. We will not recall their general definitions here. We only note that for $\emptyset \neq \mathcal{A} \subset C(K, \mathbb{F})$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the system \mathcal{A} is a ring if \mathcal{A} is a subspace closed under multiplication. The system \mathcal{A} is self-adjoint if $\bar{f} \in \mathcal{A}$ whenever $f \in \mathcal{A}$. A system $\mathcal{A} \subset C(K, \mathbb{R})$ is a lattice if it is closed under max and min of two (respectively finitely many) functions.

Věta 46 (Stone–Weierstrass Theorem). *Let K be a compact topological space.*

- (a) *Let $\mathcal{B} \subset C(K, \mathbb{R})$ be a vector subspace containing the constants and separating points. If \mathcal{B} is a ring or a lattice, then \mathcal{B} is dense in $C(K, \mathbb{R})$.*
- (b) *Let $\mathcal{B} \subset C(K, \mathbb{C})$ be a vector subspace containing the constants and separating points, which is moreover self-adjoint. If \mathcal{B} is a ring, then \mathcal{B} is dense in $C(K, \mathbb{C})$.*

Důkaz. The proof was presented, it will be examined. □

Corollary 47. *Polynomials with rational coefficients form a dense subset of the space $C([0, 1])$.*

Důkaz. The proof was presented, it will be examined. □

Proposition 48. *Let K be a Hausdorff compact space. Then the following statements are equivalent.*

- (a) *K has countable weight.*
- (b) *There exist countably many functions $\mathcal{A} \subset C(K)$ that separate the points of K .*
- (c) *$C(K)$ is separable.*

In particular, if K is a metrizable compact space, then $C(K)$ is separable.

Důkaz. The proof was presented, it will be examined. □

Remark. Later we will see that a Hausdorff compact space has countable weight if and only if it is metrizable.

3.3. Compactifications

Definition. Let X be a topological space. A pair (j, Y) is called a *compactification* of the space X if $j: X \rightarrow Y$ is an embedding of X onto a dense subset of a compact Hausdorff space Y . Often the compactification is identified with the space Y itself, and the mapping j is understood as an inclusion. We say that a compactification (j_1, Y_1) is *larger* than a compactification (j_2, Y_2) if there exists a continuous mapping $f: Y_1 \rightarrow Y_2$ such that $f \circ j_1 = j_2$. We call compactifications (j_1, Y_1) and (j_2, Y_2) of the space X *equivalent* if there exists a homeomorphism $h: Y_1 \rightarrow Y_2$ extending id_X (more formally, extending $i_2 \circ i_1^{-1}$).

Example. The closed interval $Y_1 = [0, 1]$ is a compactification of the open interval $X = (0, 1)$, where $j_1: (0, 1) \rightarrow [0, 1]$ is an embedding. The circle Y_2 can naturally be regarded as a compactification of X . Formally, $j_2(x) = \exp(2\pi ix)$ and Y_2 is the unit circle in the complex plane. The compactification (j_1, Y_1) is larger than the compactification (j_2, Y_2) .

Remark. A compactification of a Hausdorff compact space X is homeomorphic to X . If one compactification is larger than another and the other is larger than the first, then they are equivalent.

Definition. A topological space X is called *locally compact* if every point has a compact neighborhood.

Proposition 49 (Alexandroff compactification). *Let (X, τ) be a Hausdorff locally compact space that is not compact and let $\infty \notin X$. Consider on the space $X \cup \{\infty\}$ the topology σ generated by the base*

$$\mathcal{B} := \tau \cup \{\{\infty\} \cup (X \setminus K) : K \subseteq X \text{ is compact}\}.$$

Then $(X \cup \{\infty\}, \sigma)$ is a compactification of the space X .

Důkaz. The proof was presented, it will be examined. □

the end of 8. lecture (24. 11. 2025)

Examples. • Every compact Hausdorff space is locally compact.

- Euclidean spaces are locally compact.
- A discrete space is locally compact.
- The space $[0, \omega_1)$ is locally compact.

Lemma 50. *Let X be a Hausdorff locally compact space. Then*

- (a) every point $x \in X$ has a neighborhood base consisting of compact sets,*
- (b) if $A \subset X$ is an open or closed subspace, then A is locally compact,*
- (c) X is an open subset in each of its compactifications.*

Moreover, a Hausdorff topological space is locally compact if and only if it is homeomorphic to an open subset of some Hausdorff compact space.

Důkaz. The proof was presented, it will be examined. □

Definition. We say that a compactification (j, bX) of a space X is one-point if $|bX \setminus X| = 1$.

Lemma 51. *A Hausdorff topological space has a one-point compactification if and only if it is locally compact and not compact. Furthermore, if X is a locally compact space that is not compact, then the one-point compactification of X is uniquely determined up to equivalence of compactifications and the one-point compactification is the smallest compactification of X .*

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. □

Definition. Let X and Y be topological spaces and let \mathcal{F} be a system of functions mapping X into Y . We say that \mathcal{F} *separates points and closed sets* if for every closed set $F \subseteq X$ and every $x \in X \setminus F$ there exists $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(F)}$.

Lemma 52 (Tychonoff embedding). *Let X be a topological space and let $\mathcal{F} = \{f_i: X \rightarrow Y_i, i \in I\}$ be a family of continuous mappings, and consider the diagonal mapping $\Delta\mathcal{F} := \Delta_I f_i: X \rightarrow \prod_{i \in I} Y_i$. If \mathcal{F} separates points, then $\Delta\mathcal{F}$ is injective. If, moreover, \mathcal{F} separates points and closed sets, then $\Delta\mathcal{F}$ is an embedding.*

Důkaz. The proof was presented, it will be examined. □

Proposition 53 (Tychonoff cube). *Let X be a $T_{3\frac{1}{2}}$ topological space. Then there exists a set I such that X embeds into $[0, 1]^I$.*

Důkaz. The proof was presented, it will be examined. □

Corollary 54. *Every $T_{3\frac{1}{2}}$ topological space has some compactification.*

Důkaz. The proof was presented, it will be examined. □

Definition. Let X be a $T_{3\frac{1}{2}}$ topological space. Let $\mathcal{F} := \mathcal{C}(X, [0, 1])$ and let $\Delta\mathcal{F}: X \rightarrow [0, 1]^{\mathcal{F}}$ be the corresponding diagonal embedding. Then the compactification $(\Delta\mathcal{F}, \overline{\Delta\mathcal{F}(X)})$ (or any equivalent one) is called the *Čech–Stone compactification* (or beta compactification) and is denoted by βX .

Věta 55 (Characterization of the beta compactification). *Let X be a $T_{3\frac{1}{2}}$ topological space and let (j, Y) be a compactification of X . Then the following conditions are equivalent.*

- (a) Y is the Čech–Stone compactification of X .
- (b) Every continuous function $f: X \rightarrow [0, 1]$ can be continuously extended to the whole space Y , i.e. there exists a continuous function $\tilde{f}: Y \rightarrow [0, 1]$ such that $\tilde{f} \circ j = f$.
- (c) Every continuous function $f: X \rightarrow Z$ into a compact Hausdorff space Z can be continuously extended to the whole space Y , i.e. there exists a continuous function $\tilde{f}: Y \rightarrow Z$ such that $\tilde{f} \circ j = f$.
- (d) Y is the largest compactification of X .

Důkaz. The proof was presented, it will be examined. □

Remark. Every bounded continuous function on a $T_{3\frac{1}{2}}$ space X can be continuously extended to a (bounded) continuous function $\beta X \rightarrow \mathbb{R}$. Thus it is possible to identify the Banach spaces ℓ_∞ and $C(\beta\mathbb{N})$.

The beta compactification of simple topological spaces can be very complicated. An example is the space $\beta\mathbb{N}$, which can, for instance, be described as the space of all ultrafilters on \mathbb{N} with a suitable topology. See the exercises for more details.

4. Metrizable and Čech Completeness

4.1. Metrizable

Lemma 56. *A topological space with a countable base is Lindelöf.*

Důkaz. The proof was presented, it will be examined. □

the end of 9. lecture (1. 12. 2025)

If (X, ρ) is a metric space and $\sigma = \min\{\rho, 1\}$, then σ is again a metric on X , which moreover generates the same topology on X as the metric ρ .

Proposition 57. *A countable product of metric spaces (X_n, ρ_n) , $n \in \mathbb{N}$, is metrizable by the metric*

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{\min\{\rho_n(x_n, y_n), 1\}}{2^n}, \quad x, y \in \prod X_n.$$

Důkaz. The proof was presented, it will be examined. □

Věta 58 (Urysohn Metrization Theorem). *Every T_3 topological space with a countable base is metrizable.*

Důkaz. The proof was presented, it will be examined. □

Remark. It follows from the proof of the previous theorem that every separable metrizable space has a metrizable compactification and can be embedded into the Hilbert cube $[0, 1]^{\mathbb{N}}$.

Corollary 59. *A compact Hausdorff space is metrizable if and only if it has countable weight.*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

Corollary 60. *A continuous image of a metrizable compact space is a metrizable compact space, provided that the image is Hausdorff.*

Důkaz. The proof was presented, it will be examined. □

4.2. Completeness in Metric Spaces

Recall that a metric space is *complete* if every Cauchy sequence is convergent.

Let I be a set. Recall that on $\ell_{\infty}(I) := \{f : I \rightarrow \mathbb{R} : f \text{ bounded}\}$ we define a metric by $\rho(f, g) := \|f - g\|_{\infty} = \sup_{i \in I} |f(i) - g(i)|$. Then $(\ell_{\infty}(I), \rho)$ is a complete metric space.

Věta 61 (Universality of $\ell_{\infty}(I)$). *Let (M, d) be a metric space. Then there exists an isometric embedding $\varphi : M \rightarrow \ell_{\infty}(M)$.*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

Corollary 62 (On completion). *Every metric space (X, ρ) has a completion, i.e. there exists a complete metric space (Y, σ) and an isometric embedding $j : X \rightarrow Y$ such that $j(X)$ is dense in Y .*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. □

Proposition 63. *Every uniformly continuous mapping from a subspace A of a metric space (X, ρ) into a complete metric space (Y, σ) can be extended to a uniformly continuous mapping from \bar{A} into Y .*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. \square

Corollary 64 (Uniqueness of completion). *Let (X, ρ) be a metric space, $(Y_1, \sigma_1), (Y_2, \sigma_2)$ complete metric spaces, and $e_1: X \rightarrow Y_1, e_2: X \rightarrow Y_2$ two isometric embeddings such that $\overline{e_1(X)}^{Y_1} = Y_1$ and $\overline{e_2(X)}^{Y_2} = Y_2$. Then there exists an isometric surjective mapping $e: Y_1 \rightarrow Y_2$ such that $e_2 = e \circ e_1$.*

Důkaz. Idea of the proof was presented, but certain parts were omitted, those can be used as an additional exercise during the exams. \square

Finally, let us recall two well-known statements from Mathematical Analysis concerning complete metric spaces.

Věta 65 (Cantor). *A metric space (X, ρ) is complete if and only if for every sequence (F_n) of nonempty closed subsets of X satisfying $F_n \subset F_{n+1}$ for every $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \text{diam } F_n = 0$, the intersection $\bigcap_{n \in \mathbb{N}} F_n$ is a singleton.*

Věta 66 (Baire Theorem). *In a complete metric space, the intersection of countably many dense open sets is again dense.*

Definition. We say that a topological space is *Baire* if the intersection of countably many dense open sets is again dense.

4.3. Complete Metrizable Spaces

Definition. A topological space is called *completely metrizable* if there exists a complete metric generating its topology.

Proposition 67. *A countable product of completely metrizable spaces is completely metrizable.*

Důkaz. The proof was presented, it will be examined. \square

Věta 68 (Completely metrizable spaces). *Let (X, ρ) be a metric space. Then the following statements are equivalent.*

- (i) X is a G_δ subset of some completion of itself.
- (ii) X is a G_δ subset of every completion of itself.
- (iii) Whenever X is embedded into a complete metric space, it is embedded there as a G_δ subset.
- (iv) The space X is completely metrizable.

Důkaz. The proof was presented, it will be examined. \square

the end of 10. lecture (8. 12. 2025)

Definition. A Tychonoff space X is called *Čech-complete* if it is a G_δ subset of βX .

Lemma 69 (On the remainder of compactifications). *If (j_1, Y_1) and (j_2, Y_2) are compactifications of a topological space X and $f: Y_1 \rightarrow Y_2$ satisfies $f \circ j_1 = j_2$, then $f(Y_1 \setminus j_1(X)) = Y_2 \setminus j_2(X)$.*

Důkaz. The proof was presented, it will be examined. \square

Proposition 70 (Characterization of Čech-complete spaces). *For a Tychonoff space X , the following conditions are equivalent.*

- (a) X is Čech-complete.
- (b) X is a G_δ subset of every compactification.
- (c) X is a G_δ subset of some compactification.

Důkaz. The proof was presented, it will be examined. □

Remark. In the previous theorem, one cannot equivalently say “ G_δ in every compact space.” For example, the space $[0, 1]^{\omega_1}$ is not a G_δ subset of $[0, 2]^{\omega_1}$. (The proof will be given in the exercises.)

Věta 71 (Frolík’s internal characterization of Čech completeness). *Let X be a Tychonoff topological space. Then the following conditions are equivalent.*

- (i) X is Čech-complete.
- (ii) There exists a sequence of open covers $\{\mathcal{U}_n\}$ of the space X such that for every centered system of closed sets \mathcal{F} with the property that for every $n \in \mathbb{N}$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}_n$ such that $F \subset U$, we have $\bigcap \mathcal{F} \neq \emptyset$.

Důkaz. The proof was presented, it will be examined. □

Corollary 72 (Čech). *A metrizable space X is completely metrizable if and only if it is Čech-complete.*

Důkaz. The proof was presented, it will be examined. □

the end of 11. lecture (15. 12. 2025)

Examples. • Every locally compact Hausdorff space is Čech-complete.

- The space of irrational numbers is Čech-complete, while the space of rational numbers is not Čech-complete.

Věta 73 (Baire Theorem for Čech-complete spaces). *Every Čech-complete space is Baire.*

Důkaz. The proof was presented, it will be examined. □

Proposition 74 (Preservation of Čech completeness under operations). *Čech completeness is preserved under*

- (i) topological sums,
- (ii) countable products,
- (iii) closed subspaces.

Důkaz. The proof was presented, it will be examined. □

Věta 75. *The product of arbitrarily many Čech-complete spaces is Baire.*

Důkaz. Proof was omitted and will not be a part of the exam. □

Remark. Čech completeness is not preserved under arbitrary products, and there therefore exist spaces that are Baire but not Čech-complete.

5. Topological Groups

This part of the syllabus was not covered in the lectures. For an overview, one may consult the prepared lecture notes. A more detailed treatment of this material will be given in the course General Topology 2.

the end of 12. lecture (5. 1. 2026)