EXERCISES - UNIFORM SPACES

1. Find two uniformities $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathbb{R}$ which both generate the usual topology on $\mathbb{R}$, yet there exists no uniform homeomorphism between $(\mathbb{R}, \mathcal{D}_1)$ and $(\mathbb{R}, \mathcal{D}_2)$.

2. Let $(X, \mathcal{U})$ be a uniform space. Prove that if $(X, \mathcal{D})$ is metrizable, then $(X, \tau_{\mathcal{D}})$ is metrizable, and find an example showing that the converse implication does not hold.

3. Uniformity given by a system of coverings: Let X be a set. A system $\mathscr{U}$ is called a *covering uniformity* on the set X if:

(a) Every $\mathcal{U} \in \mathscr{U}$ is a covering of X.

(b) If $\mathcal{U} \in \mathscr{U}$ and $\mathcal{U}$ is a refinement of some covering $\mathcal{V}$, then $\mathcal{V} \in \mathscr{U}$.

(c) For $\mathcal{U}, \mathcal{V} \in \mathscr{U}$ it holds that $\mathcal{U} \wedge \mathcal{V} \in \mathscr{U}$.

(d) For every $\mathcal{U} \in \mathscr{U}$ there exists $\mathcal{V} \in \mathscr{U}$ such that $\mathcal{V}$ strongly star-refines $\mathcal{U}$.

(If X is a set and $\mathcal{U}$ is a covering of X, then a system $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ if $\mathcal{V}$ is a covering of X and for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. For a system $\mathcal{S}$ of subsets of X and $A \subseteq X$ define $st_{\mathcal{S}}(A) = \bigcup\{S \in \mathcal{S} \colon S \cap A \neq \emptyset\}$. We say that a covering $\mathcal{U}$ *strongly star-refines* $\mathcal{V}$ if $\{st_{\mathcal{U}}(U) \colon U \in \mathcal{U}\}$ refines $\mathcal{V}$. For two coverings $\mathcal{U}, \mathcal{V}$ we define their common refinement $\mathcal{U} \wedge \mathcal{V}$ as $\{U \cap V \colon U \in \mathcal{U}, V \in \mathcal{V}\}$.)

If $\mathscr{U}$ is a covering uniformity, set

$$\mathcal{D}_{\mathscr{U}} = \{D \subseteq X \times X \colon \exists \mathcal{U} \in \mathscr{U} \colon \bigcup\{U \times U \colon U \in \mathcal{U}\} \subseteq D\},$$

and if $\mathcal{D}$ is a uniformity, set

$$\mathscr{U}_{\mathcal{D}} = \{\mathcal{U} \in \mathcal{P}(\mathcal{P}(X)) \colon \exists D \in \mathcal{D} : \{D[x] : x \in X\} \text{ refines } \mathcal{U}\}.$$

Prove that the mapping $\mathcal{U} \mapsto \mathcal{D}_{\mathcal{U}}$ is a bijection between covering uniformities and uniformities, where the inverse mapping is given by $\mathcal{D} \mapsto \mathcal{U}_{\mathcal{D}}$.

4. Topology on a subspace and product: Prove that if $(X, \mathcal{D})$ is a uniform space and $A \subset X$, then the topology on A corresponding to $\mathcal{D}|_A$ coincides with the subspace topology. Furthermore, prove that if $(X_i, \mathcal{D}_i)$, $i \in I$ are uniform spaces, then the topology on $\Pi_I X_i$ corresponding to $\mathcal{D}_{\Pi_I X_i}$ is the product topology.

5. Existence of completion: Let $(X, \mathcal{D})$ be a T_1 uniform space where $\mathcal{D}$ is generated by a system of pseudometrics R.

- Denote $\widetilde{X}_0 := \{(x_i)_{i \in I} \colon (x_i) \text{ is a Cauchy net}\}$. Whenever $d \in R$, consider $\widetilde{d}_0 : \widetilde{X}_0 \times \widetilde{X}_0 \to [0, \infty)$ defined by
$$\widetilde{d}_0\big((x_i), (y_j)\big) := \lim_{(i,j) \in I \times J} d(x_i, y_j), \qquad \text{for } (x_i)_{i \in I} \in \widetilde{X}_0 \text{ and } (y_j)_{j \in J} \in \widetilde{X}_0$$
(on $I \times J$ we consider the ordering $(i, j) \leq (i', j')$ iff $i \leq i'$ and at the same time $j \leq j'$).
Prove that $\widetilde{d}_0$ is a well-defined pseudometric on $\widetilde{X}_0$.

- For $x \in \widetilde{X}_0$ consider $[x] := \{y \in \widetilde{X}_0 \colon \widetilde{d}_0(x, y) = 0 \text{ for every } d \in R\}$. Consider on the space $\widetilde{X} := \{[x] \colon x \in \widetilde{X}_0\}$ the system of pseudometrics $\widetilde{R} := \{\widetilde{d} \colon d \in R\}$, where for $d \in R$ the pseudometric $\widetilde{d}$ is given by $\widetilde{d}([x], [y]) := \widetilde{d}_0(x, y)$. Prove that $\widetilde{R}$ is a well-defined system of pseudometrics on $\widetilde{X}$ and that $(\widetilde{X}, \widetilde{R})$ is a T_1 uniform space.

- Define a mapping $e : X \to \widetilde{X}$ by $e(x) := [x]$ (the point x is identified with the net consisting of a single point). Prove that e is a uniform embedding and that $e(X)$ is dense in $\widetilde{X}$.

- Prove that $\widetilde{X}$ is complete.
(*Hint: it is useful to first realize that if D is a dense subset of a uniform space M, then M is complete if and only if every Cauchy net consisting of points in D converges in M.*)

EXERCISES - TOPOLOGICAL GROUPS

1. The topological group $H(K)$ – basics: Let K be a T_2 compact space. a) Prove that $H(K)$ is a topological group. (Here the symbol $H(K)$ denotes the set of all surjective homeomorphisms $f : K \to K$ with the group operation of composition and the *compact-open* topology, i.e. the topology whose subbasis consists of sets $E[L; U] := f \in H(K)\colon f(L) \subset U$, where $L \subset K$ is compact and $U \subset K$ is open.) b) Prove that if (K, d) is a metric compact space, then the compact-open topology on $H(K)$ is metrizable by the metric ρ defined by $\rho(f, g) := \sup_{x \in K} d(f(x), g(x))$ for $f, g \in H(K)$.

2. Topology on a group generated by a system of bi-invariant pseudometrics. Let G be a group and let R be a system of bi-invariant pseudometrics separating points. Then the *topology generated by the system* R is the topology τ_R, whose neighborhood basis at a point x has a subbasis generated by $B_\rho(x, \varepsilon)\colon \varepsilon > 0, ; \rho \in R$. a) Prove that G with the topology τ_R is a T_1 topological group which is SIN. b) Prove that a T_1 topological group $(G, \cdot, \tau)$ is SIN if and only if there exists a system R of bi-invariant pseudometrics satisfying $\tau = \tau_R$.

3. Topological groups that are / are not SIN: a) Consider the topological group $G = (a, b)\colon a > 0, b \in \mathbb{R}$ with the group operation defined by $(a, b) \cdot (a', b') := (aa', b + ab')$ and with the topology inherited from $\mathbb{R}^2$. Prove that G is a metrizable topological group which is not metrizable by a bi-invariant metric. (*Hint for proving that it is not metrizable by a bi-invariant metric: For a contradiction assume that G is metrizable by a bi-invariant metric ρ. Show that then there exists $\varepsilon > 0$ and $t \neq 0$ satisfying $(1, t) \in B_\rho(e, \varepsilon) \subset (a, b)\colon |b| < 1$, deduce that then $(1, at) \in B_\rho(e, \varepsilon)$ for every $a > 0$, which yields a contradiction.*)

b) Decide and prove whether the metrizable topological group $H([0, 1])$ is SIN. (For the definition and basic properties of $H([0, 1])$ see Example 1 above).

c) Consider the group S_∞ (permutations of $\mathbb{N}$ with the operation of composition) with the topology of pointwise convergence (i.e. basic neighborhoods of $\pi \in S_\infty$ consist of sets $U[\pi; F] := \sigma\colon \sigma|F = \pi|F$, where $F \subset \mathbb{N}$ are finite subsets). Prove that then $S\infty$ is a metrizable topological group. Decide and prove whether $S\infty$ is SIN.

4. Closed subgroups and quotients of $\mathbb{R}$: On $\mathbb{R}$ consider the operation $+$ and the standard topology generated by the absolute value. a) Find all closed subgroups of $\mathbb{R}$. b) Show that every quotient of $\mathbb{R}$ by a closed subgroup is, as a topological group, isomorphic to 0, $\mathbb{R}$, or $\mathbb{T}$. ($\mathbb{T}$ denotes the circle $e^{ix}\colon x \in \mathbb{R} \subset \mathbb{C}$ with the operation $e^{ix}e^{iy} = e^{i(x+y)}$ and the topology inherited from $\mathbb{C} = \mathbb{R}^2$).

5. Find an example of a T_1 topological group which is not T_4.

EXERCISES - PARACOMPACTNESS

1. Preservation of paracompactness:

(a) Prove that an F_σ subset of a paracompact space is paracompact.

(b) Find an example of a subset of a paracompact space which is not paracompact.

(c) Find an example of a paracompact space X such that $X \times X$ is not paracompact.

(d) Prove that the product of a paracompact space and a compact T_2 space is paracompact.

(e) A topological space is called σ-compact if it is the union of countably many of its compact subspaces. Prove that the product of a paracompact and a σ-compact $T_{3\frac{1}{2}}$ space is paracompact. (Observe that as a consequence we obtain that for every Banach space X, the space (X^*, w^*) is paracompact, since (X^*, w^*) is σ-compact.)

(f) Prove that $[0, \omega_1)$ is collectionwise normal but not paracompact. (*Hint:* Prove that in a countably compact space every locally finite cover is finite. Deduce that every paracompact space which is countably compact is already compact. However, $[0, \omega_1)$ is countably compact and not compact.)

2. Metric spaces embed into a countable product of hedgehogs: For a set I define the *metric hedgehog with I spines* as the metric space $(I \times [0,1])/\sim$, where $(i, x) \sim (j, y)$ iff $x = y = 0$, equipped with the metric

$$d((i,x),(j,y)) := \begin{cases} |x-y| & i = j, \\ x+y & i \neq j. \end{cases}$$

Prove that every metric space embeds into a countable product of metric hedgehogs.
(*Hint: try to be inspired by the proof of the Bing–Nagata–Smirnov theorem*)

EXERCISES - CONNECTEDNESS

1. Cardinality of a connected space: a) Prove that a connected $T_{3\frac{1}{7}2}$ space with at least two points has cardinality at least continuum. b) Prove that a connected T_3 space with at least two points is uncountable. c) Find an example of an infinite connected T_2 space which is countable.
(*Hint: first find a system of pairwise disjoint subsets of rational numbers* $\mathbb{Q}_n$, $n \in \mathbb{N} \cup \{0\}$ *such that each* $\mathbb{Q}_n$ *is dense in* $\mathbb{R}$. *Then consider* $X = \{\infty\} \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathbb{Q}_n \times \{n\}$, *where a neighborhood basis of* ∞ *consists of sets* $\{\infty\} \cup \bigcup_{n \geq k} \mathbb{Q}_n \times \{n\}$ *for* $k \in \mathbb{N}$, *a neighborhood basis of* (q,k) *for* $q \in \mathbb{Q}_k$ *and* k *odd is given by* $X \cap \big((q-\varepsilon, q+\varepsilon) \times \{k-1, k, k+1\}\big)$ *and a neighborhood basis of* (q,k) *for* $q \in \mathbb{Q}_k$ *and* k *even is given by* $X \cap \big((q-\varepsilon, q+\varepsilon) \times \{k\}\big)$. *Verify that then* X *is countable,* T_2, *and whenever a clopen set* $C \subset X$ *contains* ∞ *then* $C = X$.)

2. Connected linearly ordered topological spaces: Prove that a linearly ordered topological space $(X, \leq)$ is connected if and only if every nonempty bounded above subset has a supremum and $(x,y) \neq \emptyset$ for every $x, y \in X$.

3. Connected components of $H([0,1])$: Describe all connected components of the topological group $H([0,1])$. (For the definition and basic properties of $H([0,1])$ see Exercise 1 on topological groups.)

4. Connected components of $GL(n,\mathbb{R})$: Describe all connected components of the topological group $GL(n,\mathbb{R})$.

5. Inverse limits and examples of continua: *An inverse sequence* is a sequence of pairs $(X_n, f_n)_{n=1}^{\infty}$, where X_n are topological spaces and $f_n : X_{n+1} \to X_n$ are continuous mappings. The *inverse limit* of the inverse sequence $(X_n, f_n)_{n=1}^{\infty}$ is the following subspace $\lim_{\leftarrow}(X_n, f_n)$ of the product $\prod_{\mathbb{N}} X_n$:

$$\lim_{\leftarrow}(X_n, f_n) := \{(x_n)_{n \in \mathbb{N}} : f_n(x_{n+1}) = x_n \text{ for each } n \in \mathbb{N}\}.$$

(Sometimes we also denote it by $\lim_{\leftarrow} X_n$ or X_∞, if it is clear from the context what X_n and f_n are.)
a) Prove that if X_n are Hausdorff compact spaces (resp. continua), then X_∞ is a Hausdorff compact space (resp. a continuum).
b) Find an inverse system consisting of finite discrete spaces whose inverse limit is homeomorphic to the Cantor discontinuum.
c) An inverse sequence $(X_n, f_n)n = 1^\infty$ is called *indecomposable* if for every $i \in \mathbb{N}$ and every two continua $A, B \subset Xi + 1$ whose union is X_{i+1}, it holds that $f_i(A) = X_i$ or $f_i(B) = X_i$. Prove that if X_n are continua and $(X_n, f_n)n = 1^\infty$ is an indecomposable inverse sequence, then $X\infty$ is an indecomposable continuum.
d) Let $X_n = [0,1]$ and $f_n(x) = |1 - 2x|$ for every $n \in \mathbb{N}$. Prove that X_∞ is then an indecomposable continuum.
(*Note: this is the so-called Knaster continuum.*)

6. Strongly zero-dimensional metric spaces: We say that a metric ρ on X is an *ultrametric* if $\rho(x,y) \leq \max \rho(x,z), \rho(z,y)$ for every $x, y, z \in X$. In that case we say that (X, ρ) is an *ultrametric space*. We say that a topological space is *ultrametrizable* if it is homeomorphic to an ultrametric space. Prove the following.
a) Every ultrametric space is zero-dimensional. (*Hint: open balls are closed sets.*)
b) The space 2^ω is ultrametrizable. Deduce from the theorem from the lecture that a separable metric space is zero-dimensional if and only if it is ultrametrizable.
The following exercises aim to show that an analogue of the previous statement also holds for nonseparable spaces. There exists an example (rather complicated)of a nonseparable metric space which is zero-dimensional and not strongly zero-dimensional. It turns out that the correct notion is that of strong zero-dimensionality.
c) A metrizable space is strongly zero-dimensional if and only if it is ultrametrizable.

7. An example of a normal subspace of a compact space which is zero-dimensional but not strongly zero-dimensional:
a) Prove that there exists a decomposition A_α: $\alpha < \omega_1$ of the interval $[0,1]$ such that for every $\alpha < \omega_1$ the set A_α is

dense in the interval $[0,1]$. b) For $\alpha < \omega_1$ denote $B_\alpha = \bigcup A_\beta \colon \beta \leq \alpha$, where the sets A_α are as above. Let $D \subseteq [0,1] \times \omega_1$ be defined by

$$D = \{(r, \alpha) \colon r \in B_\alpha, \alpha < \omega_1\}.$$

Prove that the space D is zero-dimensional.
c) Let $D \subset [0,1] \times [0, \omega_1]$ be as above. Denote $D^* := D \cup ([0,1] \times \omega_1)$. Prove that D^* is a normal space.
d) Let D and D^* be as above. Prove that whenever A, B are disjoint closed subsets of D, then their closures in D^* are also disjoint.
e) Let $D \subset [0,1] \times [0, \omega_1]$ be as above. Using all the information above prove that D is a normal space which is zero-dimensional but not strongly zero-dimensional.

Let X be T_4. Recall that $A \subset X$ is a *zero* (resp. *cozero*) set if there exists a continuous function $f : X \to \mathbb{R}$ satisfying $A = f^{-1}(0)$ (resp. $A = f^{-1}(\mathbb{R} \setminus 0)$). We say that $M \subseteq X$ is C^**-embedded* if every continuous function $f : M \to [0,1]$ has a continuous extension to a continuous function $\widetilde{f} : X \to [0,1]$.

1. Dimension of a subspace:
a) Using the previous exercises find an example of a compact space X and its normal subspace M such that $\operatorname{Ind} M > \operatorname{Ind} X$ and also $\dim M > \dim X$.
b) Let X and M be T_4 spaces and $M \subset X$ be C^*-embedded. Prove that then $\dim M \leq \dim X$.
c) Let D and D^* be as in the Exercises on Connectedness, Example 7. Prove that $D \subset D^*$ is C^*-embedded and using the previous result deduce that $\dim D \leq \dim D^*$.
d) Let D and D^* be as in the Exercises on Connectedness, Example 7. Prove that $\operatorname{ind} D = 0 < 1 = \dim D \leq \operatorname{Ind} D$. (Automatically we also obtain $1 = \operatorname{ind} D^* = \dim D^* = \operatorname{Ind} D^*$.)

2. Dimension of certain dense subsets: Let X be T_4.
a) Prove that for every finite open cover $G_1, \ldots, G_k$ there exist covers $\mathcal{H} = H_1, \ldots, H_k$ and $\mathcal{E} = E_1, \ldots, E_k$ such that $\mathcal{H}$ consists of cozero sets, $\mathcal{E}$ consists of zero sets and $H_i \subset E_i \subset G_i$, $i = 1, \ldots, k$.
b) Prove that whenever $M \subset X$ is dense and C^*-embedded, then for every open cover $U_1, \ldots, U_k$ of M consisting of cozero sets in M there exists an open cover $V_1, \ldots, V_k$ of X satisfying $V_i \cap M = U_i$ for every $i = 1, \ldots, k$.
c) Prove that whenever $M \subset X$ is dense, C^*-embedded and normal, then $\dim M \geq \dim X$.
(Using Example 1b) above we even obtain $\dim M = \dim X$, in particular we get that $\dim X = \dim \beta X$.)

3. The dimension ind **can be much smaller than** dim**:**
a) Let X be a separable metric space. Prove that there exists a cover $X_\alpha \colon \alpha < \omega_1$ of the space X such that $X_\alpha \subset X_\beta$ for $\alpha < \beta$ and $\dim X_\alpha \leq 0$ for every $\alpha < \omega_1$.
(*Hint: use that* $X \subset [0,1]^\omega$ *and thus it suffices to prove the statement for* $X = [0,1]^\omega$*. You may use that for the interval* $[0,1]$ *the corresponding decomposition has already been constructed in the Exercises on Connectedness, Example 7a).*)
b) Recall that in the Exercises on Connectedness, Example 7, using the interval $[0,1]$ we constructed a normal space D for which in Example 1 above we proved that $\operatorname{ind} D = 0 < 1 = \dim D$. Try to devise how to perform an analogue of this construction and thus construct an example of a normal space D satisfying $\operatorname{ind} D = 0 < \infty = \dim D = \operatorname{Ind} D$.
(*Hint: instead of the interval* $[0,1]$ *in the construction work with* $[0,1]^\omega$*, to prove that the corresponding constructed set* D *satisfies* $\dim D = \infty$ *argue analogously as in Example 1c) above: show that* D *is* C^**-embedded in* D^* *and thus using Example 2 above conclude that* $\dim D = \dim D^* \geq \dim [0,1]^\omega = \infty$.
Note: if we carried out the construction with $[0,1]^n$*, we would obtain a normal space* D *satisfying* $\operatorname{ind} D = 0 < n = \dim D = \operatorname{Ind} D$.)

4. Small inductive dimension in compact spaces:
a) Prove that for every T_2 compact space X it holds that $\operatorname{ind} X = 1$ if and only if $\operatorname{Ind} X = 1$.
b) Let X and Y be T_2 compact spaces satisfying $\operatorname{Ind}(X) \leq 1$ and $\operatorname{Ind}(Y) \leq 0$. Show that then $\operatorname{Ind}(X \times Y) \leq 1$.
c) Prove that if a T_4 space X is the union of closed subsets A and B, then $\operatorname{ind}(X) \leq \operatorname{ind}(A) + \operatorname{ind}(B)$.
d) Let $L_0 = [0, \omega_1) \times [0,1)$ be the linearly ordered space ordered lexicographically and let $L := L_0 \cup \omega_1$ be the linearly ordered space where $x \leq \omega_1$ for $x \in L$. Prove that if on L we consider the topology given by the order, then L is a T_2 compact space satisfying $\dim L = \operatorname{ind} L = \operatorname{Ind} L = 1$. Note: the space L is called the *long line segment.*

5. The dimension dim **can be smaller than** ind**:**
a) *Construction of two Cantor functions:* On the Cantor discontinuum 2^ω consider the lexicographic order, by the symbol $0 \in 2^\omega$ and $1 \in 2^\omega$ denote the largest and smallest element in the lexicographic order, for $s \in 2^n$ denote by $x_s \in 2^\omega$ the point such that $x_s|1, \ldots, n = s$ and $x_s(i) = s(n)$ for $i \geq n$, for $s \in 2^n$ satisfying $s(n-1) = 0$ and $s(n) = 1$ denote by $s^+ \in 2^n$ the sequence given by $s|1, \ldots, n-2 = s^+|1, \ldots, n-2$, $s^+(n-1) = 1$ and $s^+(n) = 0$ and consider the countable subset $D := x_s \colon s \in 2^n, ; n \in \mathbb{N}$. Find continuous surjections $h_i : 2^\omega \to [0,1]$ for $i = 1, 2$ such that they

are nonincreasing (i.e. whenever $x < y$ then $h_i(x) \leq h(y)$), $h_i(x) = h_i(y)$ if and only if $x, y = x_s, xs^+$ for some $s \in D$, $0 = h_1(0) = h_2(0)$, $1 = h_1(1) = h_2(1)$ and finally $h_1(D) \setminus 0, 1$ and $h_2(D) \setminus 0, 1$ are disjoint dense subsets of the interval $[0, 1]$.

Let L be the long line segment from Exercise 4d) above and $h_i : 2^\omega \to [0, 1]$ for $i = 1, 2$ be as above. Consider the equivalence E on the space $(L \times 0, 1) \times 2^\omega$ defined by xEy if $x = y$ or $x, y \in ((\omega_1, 0) \times h_1^{-1}(t)); \cup; ((\omega_1, 1) \times h_2^{-1}(t))$ for some $t \in [0, 1]$. Denote by S the corresponding quotient and $q : (L \times 0, 1) \times 2^\omega \to S$ the corresponding quotient mapping. Further denote $I := q((\omega_1, 0) \times 2^\omega)$ and $S_i := q(L \times i \times 2^\omega)$ for $i = 0, 1$. Prove the following.

b) S is a compact T_2 space, $S = S_1 \cup S_2$, $S_1 \cap S_2 = I$, I is homeomorphic to $[0, 1]$, the restriction of q to $\big((L \setminus \omega_1) \times 0\big) \times 2^\omega$ is a homeomorphism onto $S_i \setminus I$ and the topology on I is induced by the linear order $q(\omega_1, 0, s) \leq q(\omega_1, 0, t)$ iff $h_1(s) \leq h_1(t)$ in $[0, 1]$.

c) Prove that $\operatorname{ind} S_1 \leq 1$ and deduce that $1 = \dim S_i = \operatorname{ind} S_i = \operatorname{Ind} S_i = \dim S$ for $i = 1, 2$.

d) Choose in S an arbitrary neighborhood G of the point $q(\omega_1, 0, 0)$ satisfying $q(\omega_1, 0, 1) \notin \overline{G}$. Prove that ∂G is not 0-dimensional and deduce that $\operatorname{ind} S = 2 \leq \operatorname{Ind} S$. (*Thus we have obtained an example of a space where* $\dim S < \operatorname{ind} S \leq \operatorname{Ind} S$ *and moreover an example showing that the addition theorem for the dimensions* ind *and* Ind *does not hold even in compact spaces.*)