1. Uniform Spaces

1.1. Relation to Topological and Metric Spaces

Remark. For a set X, denote the diagonal by $\Delta(X) = \{(x,y) \in X^2 \colon x = y\}$. The inverse relation to a relation E is the relation $\{(x,y) \colon (y,x) \in E\}$. For binary relations C, D on a set X we define $C \circ D = \{(x,z) \in X \times X \colon (x,y) \in C, (y,z) \in D \text{ for some } y \in X\}$.

Definition. A pair $(X, \mathcal{D})$ is called a *uniform space* (US) if X is a set and $\mathcal{D} \subseteq \mathcal{P}(X \times X)$ is nonempty and satisfies:

(Ua) $\forall D \in \mathcal{D} \colon \ \Delta(X) \subseteq D$,

(Ub) $\forall C, D \in \mathcal{D} \colon \ C \cap D \in \mathcal{D}$,

(Uc) if $D \in \mathcal{D}$ and $D \subseteq E$, then $E \in \mathcal{D}$,

(Ud) $\forall D \in \mathcal{D} \colon \ D^{-1} \in \mathcal{D}$,

(Ue) $\forall D \in \mathcal{D} \ \exists C \in \mathcal{D} \colon \quad C \circ C \subseteq D$.

The system $\mathcal{D}$ is called a *uniformity*. Elements of $\mathcal{D}$ are called *neighborhoods of the diagonal*. A uniformity $\mathcal{D}$ is called separated if, in addition,

(Uf) $\bigcap \mathcal{D} = \Delta(X)$. (Equivalently, for every $x, y \in X$, $x \neq y$, there exists $D \in \mathcal{D}$ such that $(x,y) \notin D$.)

If $\mathcal{D}$ is separated, then we say that the uniform space $(X, \mathcal{D})$ is T_1.

A system $\mathcal{B} \subseteq \mathcal{P}(X^2)$ is called a *base of a uniformity* (respectively, a *base of the uniformity* $\mathcal{D}$) if closing $\mathcal{B}$ under supersets yields a uniformity (respectively, the uniformity $\mathcal{D}$). A system $\mathcal{S} \subseteq \mathcal{P}(X^2)$ is called a *subbase of a uniformity* (respectively, a *subbase of the uniformity* $\mathcal{D}$) if closing it under finite intersections yields a base of a uniformity (respectively, a base of the uniformity $\mathcal{D}$).

If $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are uniform spaces, then we say that a mapping $f \colon (X, \mathcal{D}) \to (Y, \mathcal{E})$ is *uniformly continuous* if $(f \times f)^{-1}(E) \in \mathcal{D}$ for every $E \in \mathcal{E}$. A mapping f is called a *uniform homeomorphism* if f is a bijection and both f and f^{-1} are uniformly continuous.

Lemma 1. *A nonempty system $\mathcal{B} \subseteq \mathcal{P}(X^2)$ forms a base of some uniformity on X if and only if the following conditions hold:*

(a) $\forall C \in \mathcal{B} \colon \ \Delta(X) \subseteq C$,

(b) $\forall C, D \in \mathcal{B} \ \exists E \in \mathcal{B} \colon \quad E \subseteq C \cap D$,

(c) $\forall C \in \mathcal{B} \ \exists D \in \mathcal{B} \colon \quad D \subseteq C^{-1}$,

(d) $\forall D \in \mathcal{B} \ \exists C \in \mathcal{B} \colon \quad C \circ C \subseteq D$.

Moreover, if $\mathcal{B}$ is a base of a uniformity, then it is a base of a separated uniformity if and only if $\bigcap \mathcal{B} = \Delta(X)$.

Důkaz. The proof is an easy exercise, it can be used as an additional exercise during the exams. □

Examples.

- The *discrete uniformity* on a set X consists of all supersets of $\Delta(X)$.
- If (X, ρ) is a pseudometric space, define $E_\rho(r) := \{(x,y) \in X \times X \colon \rho(x,y) < r\}$ for $r > 0$. Then $\{E_\rho(r) \colon r > 0\}$ is a base of a uniformity on X, which is a base of a separated uniformity if, in addition, ρ is a metric. This uniformity is denoted by $\mathcal{D}_\rho$. We say that a uniformity $\mathcal{D}$ is *metrizable* if there exists a metric ρ such that $\mathcal{D} = \mathcal{D}_\rho$.

  It is easy to verify that if (X, ρ) and (Y, σ) are metric spaces, then a mapping $f : (X, \rho) \to (Y, \sigma)$ is uniformly continuous if and only if it is uniformly continuous as a mapping between the uniform spaces $(X, \mathcal{D}_\rho)$ and $(Y, \mathcal{D}_\sigma)$.

Definition. If R is a system of pseudometrics on a set X, then the *uniformity generated by R* (denoted by $\mathcal{D}_R$) is the uniformity whose subbase is $\{E_\rho(r) \colon r > 0, \ \rho \in R\}$.

Remark. It is easy to see that if R is a system of pseudometrics on a set X, then $\mathcal{D}_R$ is separated if and only if R separates points of X (i.e. $\forall x \neq y \ \exists \rho \in R: \ \rho(x,y) > 0$). Furthermore, if S is a system of pseudometrics on a set Y, then a mapping $f: (X, \mathcal{D}_R) \to (Y, \mathcal{D}_S)$ is uniformly continuous if and only if

$$\forall \rho \in S \ \forall \varepsilon > 0 \ \exists \sigma \in R \ \exists \delta > 0 \ \forall x, y \in X: \quad \sigma(x,y) < \delta \implies \rho(f(x), f(y)) < \varepsilon.$$

Notation. For $E \subset X \times X$ and $x \in X$ we denote $E[x] := \{y \in X\colon (x,y) \in E\}$.

Proposition 2. *If $(X, \mathcal{D})$ is a uniform space, then*

$$\tau_{\mathcal{D}} = \{A \subseteq X\colon \forall x \in A \quad \exists D \in \mathcal{D}\colon D[x] \subseteq A\}$$

is a topology on X. Moreover, the following hold.

(a) If $\mathcal{B}$ is a base of the uniformity $\mathcal{D}$, then $\mathcal{B}(x) := \{D[x]\colon D \in \mathcal{B}\}$, $x \in X$, are neighborhood bases at points in $(X, \tau_{\mathcal{D}})$.

(b) $\mathcal{D}$ is separated if and only if $(X, \tau_{\mathcal{D}})$ is T_1.

(c) If $(Y, \mathcal{E})$ is a uniform space and $f: (X, \mathcal{D}) \to (Y, \mathcal{E})$ is uniformly continuous, then $f: (X, \tau_{\mathcal{D}}) \to (Y, \tau_{\mathcal{E}})$ is continuous.

(d) If $\mathcal{D}$ is generated by a system of pseudometrics R, then for every net $(x_i)_{i \in I}$ and every $x \in X$ we have that $x_i \xrightarrow{\tau_{\mathcal{D}}} x$ if and only if $\rho(x_i, x) \to 0$ for every $\rho \in R$.

Důkaz. The proof was presented, it can be examined. □

Definition. A topological space (X, τ) is called *uniformizable* if there exists a uniformity $\mathcal{D}$ such that $\tau = \tau_{\mathcal{D}}$.

Lemma 3 (On a pseudometric)**.** *Let $(X, \mathcal{D})$ be a uniform space and let $\{D_n\colon n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{D}$ satisfy*

(i) $D_0 = X \times X$,

(ii) $\forall n \in \mathbb{N}: \ D_n = (D_n)^{-1}$,

(iii) $\forall n \in \mathbb{N}: \ D_{n+1} \circ D_{n+1} \circ D_{n+1} \subseteq D_n$.

Then there exists a pseudometric ρ on X satisfying

(a) $\forall n \geqslant 1: \quad \{(x,y)\colon d(x,y) < 2^{-n-1}\} \subseteq D_n \subseteq \{(x,y)\colon d(x,y) \leqslant 2^{-n}\}$,

(b) $\mathcal{D}_\rho \subset \mathcal{D}$ and $\rho \leqslant 1$.

Důkaz. The proof was presented, it can be examined. □

Corollary 4. *Every uniformity is generated by some system of pseudometrics. Every T_1 uniformity is generated by a system of pseudometrics that separates points.*

Důkaz. The proof was presented, it can be examined. □

Theorem 5. *A T_1 uniform space is metrizable if and only if it has a countable base.*

Důkaz. The proof was presented, it can be examined. □

Remark. A countable base of $\mathcal{D}$ is something different from a countable base of $\tau_{\mathcal{D}}$. An example is the discrete uniformity whose base consists of a single element $\Delta(X)$, but $\tau_{\mathcal{D}}$ is the discrete topology and hence the weight of $(X, \tau_{\mathcal{D}})$ equals $|X|$.

Theorem 6. *A T_1 topological space is uniformizable if and only if it is $T_{3\frac{1}{2}}$.*

Důkaz. The proof was presented, it can be examined. □

1.2. Subspace, Sum and Product

Definition.

- Let $(X, \mathcal{D})$ be a uniform space and $A \subset X$. Define $\mathcal{D}|_A := \{D \cap (A \times A) \colon D \in \mathcal{D}\}$. Then $(A, \mathcal{D}|_A)$ is a *subspace* of $(X, \mathcal{D})$.
- Let $(X_i, \mathcal{D}_i)$ be uniform spaces. The *product* of uniformities is the uniformity $\mathcal{D}_{\Pi_I X_i}$ on $\Pi_I X_i$ whose subbase is $\{(\pi_i \times \pi_i)^{-1}(D) \colon i \in I, D \in \mathcal{D}_i\}$. Then $(\Pi_I X_i, \mathcal{D}_{\Pi_I X_i})$ is the *product of uniform spaces.*
- Let $(X_i, \mathcal{D}_i)$ be uniform spaces. On $\biguplus_I X_i := \bigcup_{i \in I}(\{i\} \times X_i)$ we define the *sum* of uniformities as the uniformity $\biguplus_I \mathcal{D}_i := \{\bigcup_{i \in I}(\{i\} \times D_i) \colon D_i \in \mathcal{D}_i$ for each $i \in I\}$. Then $(\biguplus_I X_i, \biguplus_I \mathcal{D}_i)$ is the *sum of uniform spaces.*

Remark. It is easy to verify that the sum/product/subspace of uniform spaces is a well-defined uniform space. The topology generated by the uniformity $\mathcal{D}|_A$ is the subspace topology, and the topology generated by the uniformity $\mathcal{D}_{\Pi X_i}$ is the product topology.

Proposition 7. *Let $(Z, \mathcal{D})$ and $(X_i, \mathcal{D}_i)$ for $i \in I$ be uniform spaces.*

(i) *A mapping $f : Z \to \Pi_I X_i$ is uniformly continuous if and only if the mappings $\pi_i \circ f$ are uniformly continuous for each $i \in I$.*

(ii) *Let $f_i : X_i \to (Y_i, \mathcal{E}_i)$, $i \in I$, be uniformly continuous mappings. Then the mapping $\Pi_I f_i : \Pi_I X_i \to \Pi_I Y_i$ is also uniformly continuous.*

(iii) *Let $f_i : Z \to X_i$, $i \in I$, be uniformly continuous mappings. Then the mapping $\Delta_I f_i : Z \to \Pi_I X_i$ is also uniformly continuous.*

(iv) *If $f, g : Z \to \mathbb{R}$ are uniformly continuous mappings, then the mappings $f+g$, $f-g$, $\max\{f,g\}$, $\min\{f,g\}$, and $|f|$ are also uniformly continuous. Moreover, if f, g are bounded functions, then the mapping $f \cdot g$ is also uniformly continuous.*

Důkaz. The proof was presented, it can be examined. $\square$

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1.3. Completeness and Total Boundedness

Definition.

- A net $(x_i)_{i \in I}$ in a uniform space $(X, \mathcal{D})$ is called *Cauchy* if for every $D \in \mathcal{D}$ there exists $i_0 \in I$ such that for all $i, j \geqslant i_0$ we have $(x_i, x_j) \in D$.
- A uniform space $(X, \mathcal{D})$ is called *complete* if every Cauchy net converges in $(X, \tau_{\mathcal{D}})$.
- A uniform space $(X, \mathcal{D})$ is called *totally bounded* if for every $E \in \mathcal{D}$ there exists a finite set $K \subseteq X$ such that $E[K] = X$. (Where $E[K] := \bigcup_{x \in K} E[x]$.)

Remark. If the uniformity $\mathcal{D}$ is generated by a system of pseudometrics R, then a net (x_i) is Cauchy in $(X, \mathcal{D})$ if and only if

$$\forall \rho \in R \; \forall \varepsilon \; \exists i_0 \; \forall i, j \geqslant i_0 : \quad \rho(x_i, x_j) < \varepsilon.$$

Remark. It is not difficult to see that in a complete metric space Cauchy nets are convergent. It then easily follows that a metric space (X, ρ) is complete (resp. totally bounded) if and only if the uniform space $(X, \mathcal{D}_\rho)$ is complete (resp. totally bounded).

Remark. Let $(X, \mathcal{D})$ be a uniform space. It is easy to verify that convergent nets in $(X, \tau_{\mathcal{D}})$ are Cauchy in $(X, \mathcal{D})$ and that uniformly continuous mappings send Cauchy nets to Cauchy nets.

Proposition 8. *Let $(X, \mathcal{D})$ be a T_1 uniform space. Then*

$$X \textit{ is totally bounded} \Leftrightarrow \textit{every net in } X \textit{ has a Cauchy subnet.}$$

Důkaz. The proof was presented, it can be examined. ☐

Proposition 9. *(i) If a subspace of a complete T_1 uniform space is complete, then it is closed.*

(ii) A subspace of a totally bounded uniform space (resp. a closed subspace of a complete uniform space) is totally bounded (resp. complete).

(iii) The product of totally bounded (resp. complete) uniform spaces is totally bounded (resp. complete).

Důkaz. The proof was presented, it can be examined. ☐

Theorem 10. *Let X be a T_1 uniform space. Then $(X, \tau_{\mathcal{D}})$ is compact if and only if $(X, \mathcal{D})$ is complete and totally bounded.*

Důkaz. The proof was presented, it can be examined. ☐

Proposition 11. *Let X and Y be T_1 uniform spaces, let Y be complete, let $A \subset X$, and let $f : A \to Y$ be uniformly continuous. Then there exists a uniformly continuous mapping $F : \overline{A} \to Y$ such that $F|_A = f$.*

Důkaz. The proof was presented, it can be examined. ☐

Definition. Let $(X, \mathcal{D})$ be a T_1 uniform space. Its *completion* is a pair (e, Y), where Y is a complete T_1 uniform space and $e : X \to Y$ is a uniform embedding onto a dense subset (i.e. $\overline{e(X)} = Y$ and $e : X \to e(X)$ is a uniform homeomorphism).

Theorem 12. *Every T_1 uniform space has a completion. Moreover, if (e, Y) and (e', Y') are completions of a T_1 uniform space X, then there exists a uniform homeomorphism $F : Y \to Y'$ such that $F \circ e = e'$.*

Důkaz. The proof was omitted, it will not be examined. ☐

1.4. Uniformity on Compact Spaces

Theorem 13. *Let (X, τ) be a compact Hausdorff space. Then there exists exactly one uniformity on X that generates the topology τ; a base of this unique uniformity is formed by the open neighborhoods of the diagonal $\Delta(X)$.*

Důkaz. The proof was presented, it can be examined. ☐

Proposition 14. *Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ be T_1 uniform spaces and let (X, τ_D) be compact. Then every continuous mapping $f\colon X \to Y$ is uniformly continuous.*

Důkaz. The proof was presented, it can be examined. ☐

2. Topological Groups

Definition. A triple $(G, \cdot, \tau)$ is called a *topological group* (TG) if $(G, \cdot)$ is a group, (G, τ) is a topological space, and the multiplication operation $\cdot \colon G \times G \to G$ (where $G \times G$ is equipped with the product topology) and the inversion operation ${}^{-1} \colon G \to G$ are continuous.

Examples. Examples of topological groups include the following.

(a) Any group with the discrete topology.

(b) Any normed linear space with addition as the group operation and the topology induced by the norm. More generally, every topological vector space is a commutative topological group.

(c) $GL(n, \mathbb{R})$ (the group of all real invertible $n \times n$ matrices). The group operation is given by matrix multiplication, and the topology is given by coordinatewise convergence (i.e. the topology inherited from the product topology on $\mathbb{R}^{n \times n}$).

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(d) $\mathrm{Iso}(X)$, where (X, ρ) is a metric space. The symbol $\mathrm{Iso}(X)$ denotes the set of all surjective isometries $f : X \to X$ with the group operation given by composition and the topology of pointwise convergence (i.e. the product topology inherited from X^X).

Důkaz. The proof was presented, it can be examined. □

(e) $\mathrm{Iso}(V)$, where $(V, \|\cdot\|)$ is a normed linear space. Here $\mathrm{Iso}(V)$ denotes the set of all surjective linear isometries $f : V \to V$ with the group operation given by composition and the topology of pointwise convergence (i.e. the product topology inherited from V^V).

(f) $H(K)$, where K is a compact Hausdorff space. The symbol $H(K)$ denotes the set of all surjective homeomorphisms $f : K \to K$ with the group operation given by composition and the *compact-open* topology, i.e. the topology whose subbase is formed by the sets $E[L; U] := \{f \in H(K) \colon f(L) \subset U\}$, where $L \subset K$ is compact and $U \subset K$ is open.
(A proof that this is indeed a topological group is left as an exercise.)

The neutral element of a topological group G is denoted by e_G (or simply e if the group G is clear from the context). Recall that a subgroup $N \subset G$ is a *normal subgroup* (denoted $N \lhd G$) if $gNg^{-1} = N$ for every $g \in G$ (not to be confused with the unrelated notion of a normal topological space). Furthermore, recall that for $N \lhd G$ we can define the *quotient group* G/N, i.e. the group whose elements are cosets of the form $xN := \{xn \colon n \in N\}$, with the group operation defined naturally by $(xN)(yN) = (xy)N$ and $(xN)^{-1} = x^{-1}N$.

For each $g \in G$ we define the left translation $L_g : G \to G$ and the right translation $R_g : G \to G$ by $L_g(h) = gh$ and $R_g(h) = hg$ for $h \in G$. A topological space X is called *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.

Lemma 15. *Let G be a topological group. Then the following hold.*

(a) The inversion mapping ${}^{-1} : G \to G$ is a homeomorphism.

(b) For every $g \in G$, the left and right translations L_g and R_g are homeomorphisms of G.

(c) G is a homogeneous space.

(d) $\forall U \in \mathcal{U}(e)\ \exists V \in \mathcal{U}(e): \quad V \cdot V^{-1} \subset U.$

(e) *If one of the sets* $A, B \subset G$ *is open, then* $A \cdot B$ *is open.*

(f) *If* $H \subset G$ *is a (normal) subgroup, then* $\overline{H}$ *is also a (normal) subgroup.*

(g) *If* $H \subset G$ *is a subgroup and* $\operatorname{Int}(H) \neq \varnothing$*, then* H *is clopen.*

(h) *The product of topological groups equipped with the product topology is a topological group.*

(i) *A homomorphism of topological groups* $f\colon G \to H$ *is continuous if and only if it is continuous at the neutral element* e_G*.*

Důkaz. The proof was presented, it can be examined. ☐

2.1. Uniformities on Topological Groups

Definition. Let G be a topological group. Then

- the *right uniformity* on G is the uniformity $\mathcal{D}_R$ whose base is given by the system $\{R_U : U \in \mathcal{U}(e)\}$, where $R_U := \{(x,y) : xy^{-1} \in U\}$ for $U \in \mathcal{U}(e)$,
- the *left uniformity* on G is the uniformity $\mathcal{D}_L$ whose base is given by the system $\{L_U : U \in \mathcal{U}(e)\}$, where $L_U := \{(x,y) : x^{-1}y \in U\}$ for $U \in \mathcal{U}(e)$.

Lemma 16. *Let* $(G, \cdot, \tau)$ *be a topological group. Then* $\tau = \tau_{\mathcal{D}_R} = \tau_{\mathcal{D}_L}$ *(i.e. the topology generated by the right uniformity is the group topology). Moreover, the mappings* $R_g : (G, \mathcal{D}_R) \to (G, \mathcal{D}_R)$ *and* $L_g : (G, \mathcal{D}_L) \to (G, \mathcal{D}_L)$ *are uniform homeomorphisms for every* $g \in G$*.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 17. *Every* T_1 *topological group is* $T_{3\frac{1}{2}}$*. Moreover, a* T_1 *topological group is metrizable if and only if it has a countable character.*

Důkaz. The proof was presented, it can be examined. ☐

Lemma 18 (On right/left invariant pseudometrics)**.** *Let* $(G, \cdot, \tau)$ *be a topological group and let* $\{U_n : n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{U}(e)$ *satisfy*

(i) $U_0 = G$,

(ii) $\forall n \in \mathbb{N}: \ U_n = (U_n)^{-1}$,

(iii) $\forall n \in \mathbb{N}: \ U_{n+1} \cdot U_{n+1} \cdot U_{n+1} \subseteq U_n$.

Then there exists a right-invariant (respectively, left-invariant) pseudometric ρ *on* G *satisfying* $\rho \leqslant 1$,

(a) $\forall n \geqslant 1: \quad B_\rho(e, 2^{-n-1}) \subseteq U_n \subseteq \overline{B}_\rho(e, 2^{-n})$,

(b) $\mathcal{D}_\rho \subset \mathcal{D}_R$ *(respectively,* $\mathcal{D}_\rho \subset \mathcal{D}_L$*).*

Moreover, if the left and right uniformities on G *coincide, then there exists a bi-invariant pseudometric* ρ *with the properties above.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 19 (Birkhoff–Kakutani)**.** *Every metrizable topological group is metrizable by a right-invariant (respectively, left-invariant) metric.*

Důkaz. The proof was presented, it can be examined. ☐

Definition. We say that a topological group is SIN (Small Invariant Neighborhoods) if there exists a base $\mathcal{B}$ of neighborhoods of e such that $gUg^{-1} \subset U$ for every $U \in \mathcal{B}$ and every $g \in G$.

Proposition 20. *A topological group is SIN if and only if its left and right uniformities coincide.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 21. *A metrizable group is SIN if and only if it is metrizable by a bi-invariant metric.*

Důkaz. The proof was presented, it can be examined. ☐

Examples. Typical examples of SIN topological groups are compact groups, discrete groups, and commutative groups. An example of a metrizable topological group that is not SIN is $GL(n, \mathbb{R})$.

Důkaz. The proof was presented, it can be examined. ☐

2.2. Quotients of Topological Groups

Theorem 22. *Let G be a topological group and let $N \lhd G$. Consider on G/H the quotient topology induced by the mapping $\pi : G \to G/H$. Then G/H is a topological group and π is an open homomorphism. Moreover, G/H is T_1 if and only if H is closed (regardless of whether G itself is T_1).*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 23. *Let G be a T_1 topological group and let $H \subset G$ be a locally compact subgroup. Then H is closed.*

Důkaz. The proof was presented, it can be examined. ☐

2.3. Representations of Topological Groups

Definition. Let G be a T_1 topological group. We say that a function $f : G \to \mathbb{R}$ is right uniformly continuous if f is uniformly continuous as a function from $(G, \mathcal{D}_R)$ to $\mathbb{R}$. The set of all bounded right uniformly continuous functions $f : G \to \mathbb{R}$ is denoted by $RUC(G)$.

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Lemma 24. *Let G be a T_1 topological group. Then a function $f : G \to \mathbb{R}$ is right uniformly continuous if and only if*

$$\forall \varepsilon > 0 \ \exists U \in \mathcal{U}(e) \ \forall u \in U \ \forall x \in G : \quad |f(ux) - f(x)| < \varepsilon.$$

If $RUC(G)$ is equipped with the norm $\|f\|_\infty := \sup_{x \in G} |f(x)|$, then $(RUC(G), \|\cdot\|_\infty)$ is a Banach space. Moreover, $RUC(G)$ separates points and closed sets in G.

Důkaz. The proof was presented, it can be examined. ☐

Theorem 25 (Teleman)**.** *Let G be a T_1 topological group. Then there exists a Banach space V such that G embeds as a topological group into* $\mathrm{Iso}(V)$*.*

Důkaz. The proof was presented, it can be examined. ☐

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3. Paracompact Spaces

Definition. If X is a set and $\mathcal{U}$ is a cover of X, then a system $\mathcal{V}$ is called a *refinement of* $\mathcal{U}$ (denoted $\mathcal{V} \prec \mathcal{U}$) if $\mathcal{V}$ is a cover of X and for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$.

Furthermore, let X be a topological space and let $\mathcal{S} \subseteq \mathcal{P}(X)$. The system $\mathcal{S}$ is called

- *locally finite* if every point of X has a neighborhood intersecting only finitely many sets from $\mathcal{S}$,
- *discrete* if every point of X has a neighborhood intersecting at most one set from $\mathcal{S}$,
- *σ-locally finite* (respectively, *σ-discrete*) if it is a countable union of locally finite (respectively, discrete) systems.

Remarks. Every (σ-)discrete system is (σ-)locally finite. The system $\{(-\frac{1}{n}, \frac{1}{n})\colon n \in \mathbb{N}\}$ is σ-discrete but not locally finite.

Fact 26 (Closure of a locally finite family)**.** *If $\mathcal{A}$ is a locally finite family in a topological space X, then $\{\overline{A}\colon A \in \mathcal{A}\}$ is locally finite and $\overline{\bigcup \mathcal{A}} = \bigcup\{\overline{A}\colon A \in \mathcal{A}\}$.*

Důkaz. The proof was presented, it can be examined. ☐

Definition. A Hausdorff topological space X is called *paracompact* if every open cover of X has a locally finite open refinement.

Examples. All compact spaces and all discrete spaces are paracompact.
(Later we will prove that every metric space is also paracompact.)

Theorem 27 (Characterization of Paracompactness)**.** *For a T_3 topological space X, the following conditions are equivalent.*

(a) X is paracompact.

(b) Every open cover of X is refined by a σ-locally finite open cover.

(c) Every open cover of X is refined by a locally finite cover.

(d) Every open cover of X is refined by a locally finite closed cover.

Důkaz. The proof was presented, it can be examined. ☐

Corollary 28. *Every Lindelöf T_3 space is paracompact.*

Důkaz. The proof was presented, it can be examined. ☐

Definition. For a system $\mathcal{S}$ of subsets of a set X and $x \in X$, define $st_{\mathcal{S}}(x) = \bigcup\{S \in \mathcal{S}\colon x \in S\}$. We say that a cover $\mathcal{V}$ *star-refines* a cover $\mathcal{U}$ (denoted $\mathcal{V} \prec_{st} \mathcal{U}$) if $\{st_{\mathcal{V}}(x)\colon x \in X\}$ refines $\mathcal{U}$.

Theorem 29 (Characterization of Paracompactness II)**.** *For a Hausdorff topological space (X, τ), the following conditions are equivalent.*

(a) Every open cover $\mathcal{U}$ of X has an open star-refinement that is a cover.

(b) There exists a uniformity $\mathcal{D}$ on X generating the topology of X (i.e. $\tau_{\mathcal{D}} = \tau$) such that for every open cover $\mathcal{U}$ of X there exists $D \in \mathcal{D}$ with $\{D[x]\colon x \in X\} \prec \mathcal{U}$.

(c) X is $T_{3\frac{1}{2}}$ and for every open cover $\mathcal{U}$ of X there exists a continuous pseudometric ρ on X such that $\{B_\rho(x, 1)\colon x \in X\} \prec \mathcal{U}$.

(d) *X is $T_{3\frac{1}{2}}$ and every open cover of X is refined by a σ-discrete open cover.*

(e) *X is paracompact.*

Důkaz. The proof was presented, it can be examined. ☐

the end of 6. lecture (26.3. 2026)

Definition. A Hausdorff topological space X is called *collectively normal* if for every discrete family $\mathcal{F}$ of closed sets there exists a family of pairwise disjoint open sets $\{U(F)\colon F \in \mathcal{F}\}$ such that $F \subset U(F)$ for every $F \in \mathcal{F}$.

Remark. If the family of closed sets is finite, then it is discrete if and only if it is disjoint. In particular, every collectively normal space is normal.

Proposition 30. *Every paracompact topological space is collectively normal, and hence normal.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 31 (Stone)**.** *Every metrizable topological space is paracompact.*

Důkaz. The proof was presented, it can be examined. ☐

Definition. Let $\mathcal{G}$ be an open cover of a space X. A family of continuous functions $\{f_i : X \to [0,1]\colon i \in I\}$ is called a *locally finite partition of unity subordinate to* $\mathcal{G}$ if the family $\{\{f_i \neq 0\}\colon i \in I\}$ is locally finite, refines $\mathcal{G}$, and

$$\sum_{i \in I} f_i(x) = 1$$

for every $x \in X$.

Theorem 32 (Partition of Unity)**.** *In a paracompact topological space, for every open cover there exists a locally finite partition of unity subordinate to this cover.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 33 (Dugundji – special case)**.** *Let K be a metrizable compact space and let $L \subset K$ be a closed subset. Then there exists a linear mapping $E : C(L) \to C(K)$ such that $Ef|_L = f$ and $\|Ef\|_\infty \leqslant \|f\|_\infty$ for every $f \in C(L)$. Moreover, $Ef \geqslant 0$ whenever $f \geqslant 0$.*

Důkaz. The proof was presented, it can be examined. ☐

the end of 7. lecture (2. 4. 2026)

Theorem 34 (Bing, Nagata, Smirnov)**.** *For a $T_{3\frac{1}{2}}$ space X, the following are equivalent.*

(a) *X is metrizable.*
(b) *X has a σ-discrete base.*
(c) *X has a σ-locally finite base.*

Důkaz. The proof was presented, it can be examined. ☐

4. Connectedness

Definition. A topological space is called *connected* if it cannot be expressed as a disjoint union of two nonempty open sets.

Remark. There are also the notions of path connectedness and arc connectedness; however, we will not discuss these notions further in this lecture.

Note that some authors (e.g. Engelking) consider the empty set to be connected, while others do not.

Proposition 35. *For a topological space X, the following conditions are equivalent.*

(a) The space X is connected.

(b) If $X = A \cup B$ and $\overline{A} \cap B = \varnothing = A \cap \overline{B}$, then $A = \varnothing$ or $B = \varnothing$.

(c) The space X contains no nontrivial clopen subset.

(d) Every continuous mapping $f\colon X \to \{0,1\}$ is constant (where $\{0,1\}$ is the two-point discrete space).

Důkaz. The proof was presented, it can be examined. □

Proposition 36. *The continuous image of a connected space is connected.*

Důkaz. The proof was presented, it can be examined. □

Proposition 37 (Union of connected sets)**.** *Let $\{C_i\colon i \in I\}$ be a family of connected subsets of a space X and suppose that one of the following conditions holds:*

(a) $\exists i_0 \in I\ \forall i \in I: \quad C_i \cap C_{i_0} \neq \varnothing$; *(b)* $\bigcap_{i\in I} C_i \neq \varnothing$.

Then $\bigcup C_i$ is connected.

Důkaz. The proof was presented, it can be examined. □

Corollary 38. *If X is a topological space, $A \subseteq X$ is connected, and $A \subseteq M \subseteq \overline{A}$, then M is connected.*

Důkaz. The proof was presented, it can be examined. □

Theorem 39. *Let X be a $T_{3\frac{1}{2}}$ topological space. Then X is connected if and only if βX is connected.*

Důkaz. The proof was presented, it can be examined. □

Theorem 40. *Let X_i, $i \in I$, be nonempty topological spaces. Then $\prod_I X_i$ is connected if and only if all spaces X_i, $i \in I$, are connected.*

Důkaz. The proof was presented, it can be examined. □

Definition. Let X be a topological space and let $x \in X$. The *connected component* of the point x is the largest connected set containing x. It is denoted by C_x.

Remark. By Proposition 37, the connected component of every point exists. If C_x and C_y are two components, then either $C_x = C_y$ or $C_x \cap C_y = \varnothing$. Thus, connected components form a partition of the space X.

Proposition 41. *If X_i, $i \in I$, are topological spaces and $x = (x_i) \in \prod_I X_i$, then*

$$C_x = \prod_I C_{x_i}.$$

(That is, the component of $x = (x_i)$ is the product of the components of the corresponding x_i, $i \in I$.)

Důkaz. The proof was presented, it can be examined. ☐

Definition. Let X be a topological space. A set Q is called the *quasicomponent* of the point x if

$$Q = \bigcap\{Z : x \in Z,\ Z \text{ is clopen}\}.$$

It is denoted by Q_x.

Remark. For every $x \in X$ we have $C_x \subseteq Q_x$. Quasicomponents are closed, since they are defined as intersections of closed sets. Moreover, quasicomponents also form a partition of the space.

Example. Let X be a subset of the plane consisting of the points $a = (0,0)$, $b = (0,1)$, and a countable family of line segments joining the points $(2^{-n}, 0)$ and $(2^{-n}, 1)$. Then $C_a = \{a\} \neq \{a, b\} = Q_a$.

Důkaz. The proof was presented, it can be examined. ☐

Lemma 42 (On intersections in compact spaces)**.** *Let X be a compact space and let $\mathcal{A}$ be a family of closed sets. If $\bigcap \mathcal{A} \subseteq U$ for some open set U, then there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{A}$ such that $\bigcap \mathcal{F} \subseteq U$.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 43. *In a compact T_2 space, components and quasicomponents coincide.*

Důkaz. The proof was presented, it can be examined. ☐

4.1. Continua

Definition. A compact, connected, nonempty T_2 space is called a *continuum.* If it consists of a single point, it is called *degenerate.*

Remark. Continuous images and arbitrary products of continua are continua.

Proposition 44. *If $\mathcal{H}$ is a family of continua closed under finite intersections, then $\bigcap \mathcal{H}$ is a continuum.*
(In particular, the intersection of a decreasing sequence of continua is a continuum.)

Důkaz. The proof was presented, it can be examined. ☐

Proposition 45 (Boundary bumping)**.** *If A is a proper closed subset of a continuum X, then every component of A intersects the boundary of A.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 46 (Sierpiński)**.** *Let X be a continuum and let X_n, $n \in \mathbb{N}$, be pairwise disjoint closed subsets whose union is X. Then $X_n = \varnothing$ for all n except one.*

Důkaz. The proof was presented, it can be examined. ☐

the end of 8. lecture (9. 4. 2026)

Definition. A continuum is called *decomposable* if it can be written as the union of two proper subcontinua. Otherwise, it is called *indecomposable.*

Example. There exists an indecomposable continuum in $\mathbb{R}^2$.

the end of 9. lecture (16. 4. 2026)

4.2. Disconnectedness

Definition. A Hausdorff topological space X is called

- *hereditarily disconnected* if all components are singletons;
- *totally disconnected* if for $x \neq y$ there exists a clopen set $Z \subseteq X$ such that $x \in Z$ and $y \notin Z$;
- *zero-dimensional* (sometimes written 0-*dim*) if it has a base consisting of clopen sets;
- *strongly zero-dimensional* (sometimes written *strongly* 0-*dim*) if for every two disjoint closed sets E, F there exists a clopen set Z such that $E \subseteq Z \subseteq X \setminus F$.

Remarks.

- The terminology is not completely uniform; we use the terminology from Engelking (different terminology is used in the lecture notes). The most important notion will be zero-dimensionality (where the terminology is uniform).
- Strong zero-dimensionality as defined here automatically implies normality, but it can also be naturally defined in a reasonable way already in Tychonoff spaces (see, for example, the lecture notes for details).

Proposition 47. *Let X be a T_2 topological space. Then*

$$X \text{ is strongly } 0\text{-dim} \implies X \text{ is } 0\text{-dim} \implies X \text{ is totally disconnected} \implies X \text{ is hereditarily disconnected.}$$

Důkaz. The proof was presented, it can be examined. ☐

Examples.

- Consider $X = \mathbb{R}^2$ with a topology τ defined as follows: the points $\mathbb{Q}^2$ are isolated, and the remaining points x have basic neighborhoods of the form $\{x\} \cup (B(x, \varepsilon) \cap \mathbb{Q}^2)$ for $\varepsilon > 0$. Then (X, τ) is a T_2 space that is hereditarily disconnected but not totally disconnected. There also exists a metrizable example, but it is considerably more complicated (see the lecture notes, Example 8.46).
- Consider the Erdős space, i.e. $E := \ell_2 \cap \mathbb{Q}^\omega$ with the topology inherited from ℓ_2. Then E is a metrizable totally disconnected space that is not zero-dimensional.
- Examples of spaces that are zero-dimensional but not normal (and hence not strongly zero-dimensional according to our definition) were already mentioned in General Topology 1 (the product of the Sorgenfrey line, or the Isbell–Mrówka space).
- In exercises we will present an example of a normal space that is zero-dimensional but not strongly zero-dimensional. There even exists a metrizable example, but it is very complicated.

Důkaz. The proof was presented, it can be examined. ☐

Theorem 48 (Disconnectedness in compact spaces)**.** *For a T_2 compact space X we have:*

$$X \text{ is strongly } 0\text{-dim} \Leftrightarrow X \text{ is } 0\text{-dim} \Leftrightarrow X \text{ is totally disconnected} \Leftrightarrow X \text{ is hereditarily disconnected.}$$

Důkaz. The proof was presented, it can be examined. ☐

Theorem 49 (Zero-dimensionality of βX)**.** *Let X be T_4. Then βX is 0-dim if and only if X is strongly 0-dim.*

Důkaz. The proof was presented, it can be examined. ☐

Proposition 50. *Let X be T_2. Then X is zero-dimensional if and only if it can be embedded into 2^I for some set I. In that case, one can choose $I = w(X)$.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 51. *Every T_2 compact space is a continuous image of a zero-dimensional compact space of the same weight.*

Důkaz. The proof was presented, it can be examined. ☐

5. Topological Dimension

Definition (Small inductive dimension: Menger, Urysohn)**.** For a T_3 space X we define its small inductive dimension inductively for $n \in \mathbb{N} \cup \{0\}$ as follows:

- We say that $\operatorname{ind} X = -1$ if and only if $X = \varnothing$.
- $\operatorname{ind} X \leqslant n$ if for every $x \in X$ and every neighborhood U of x there exists an open set V such that $x \in V \subseteq U$ and $\operatorname{ind}(\partial V) \leqslant n - 1$.
- $\operatorname{ind} X = n$ if $\operatorname{ind} X \leqslant n$ and $\operatorname{ind} X \leqslant n - 1$ does not hold.
- $\operatorname{ind} X = \infty$ if $\operatorname{ind} X \leqslant n$ does not hold for any $n \in \mathbb{N}$.

We call $\operatorname{ind} X$ the *small inductive dimension* of the space X.

Remarks. Let X be a T_3 space. Then:

- $\operatorname{ind} X \leqslant 0$ if and only if X is zero-dimensional;
- if $M \subset X$, then $\operatorname{ind} M \leqslant \operatorname{ind} X$;
- $\operatorname{ind}[0, 1] = 1$.

Definition (Large inductive dimension: Brouwer, Čech)**.** For a T_4 space X we define its large inductive dimension inductively for $n \in \mathbb{N} \cup \{0\}$ as follows:

- We say that $\operatorname{Ind} X = -1$ if and only if $X = \varnothing$.
- $\operatorname{Ind} X \leqslant n$ if for every closed set E and every open set $U \supseteq E$ there exists an open set V such that $E \subseteq V \subseteq U$ and $\operatorname{Ind}(\partial V) \leqslant n - 1$.
- $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leqslant n$ and $\operatorname{Ind} X \leqslant n - 1$ does not hold.
- $\operatorname{Ind} X = \infty$ if $\operatorname{Ind} X \leqslant n$ does not hold for any $n \in \mathbb{N}$.

We call $\operatorname{Ind} X$ the *large inductive dimension* of the space X.

Remarks. Let X be a T_4 space. Then:

- if $M \subset X$ is closed, then $\operatorname{Ind} M \leqslant \operatorname{Ind} X$;
- $\operatorname{Ind} X \leqslant 0$ if and only if X is strongly zero-dimensional;
- $\operatorname{ind} X \leqslant \operatorname{Ind} X$;
- $\operatorname{Ind}[0, 1] = 1$.

Definition. We say that a family $\mathcal{A}$ of subsets of a set X has order n if n is the largest natural number for which there exist distinct elements $A_1, \dots, A_{n+1} \in \mathcal{A}$ such that $\bigcap A_i \neq \varnothing$.

Definition (Covering dimension: Čech, Lebesgue)**.** For a T_4 space X we define its covering dimension inductively for $n \in \mathbb{N} \cup \{0\}$ as follows:

- $\dim \varnothing = -1$.
- $\dim X \leqslant n$ if every finite open cover of X is refined by a finite open cover of order at most n.
- $\dim X = n$ if $\dim X \leqslant n$ and $\dim X \leqslant n - 1$ does not hold.
- $\dim X = \infty$ if $\dim X \leqslant n$ does not hold for any n.

We call $\dim X$ the *covering dimension* of the space X.

Remarks. Let X be a T_4 space. Then:

- if $M \subset X$ is closed, then $\dim M \leqslant \dim X$;
- $\dim[0, 1] = 1$.

the end of 10. lecture (23. 4. 2026)

Proposition 52. *In a* T_4 *topological space* X*, we have* $\dim X \leqslant 0$ *if and only if* X *is strongly* 0*-dimensional.*

Důkaz. The proof was presented, it can be examined. ☐

Definition. Let X be a set and let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a cover of X. An indexed family $\{T_S \colon S \in \mathcal{S}\}$ is called a *shrinking* of the family $\mathcal{S}$ if it is a cover and $T_S \subseteq S$ for each $S \in \mathcal{S}$.

Lemma 53 (On shrinking)**.** *Let* X *be a* T_4 *space and let* $\{G_1, \dots, G_n\}$ *be an open cover of* X*. Then there exists an open cover* $\{H_1, \dots, H_n\}$ *of* X *such that* $\overline{H_i} \subset G_i$ *for* $i \in \{1, \dots, n\}$*.*
(That is, every finite open cover has a closed shrinking whose interiors also form a cover.)

Důkaz. The proof was presented, it can be examined. ☐

the end of 11. lecture (30. 4. 2026)

Lemma 54 (On swelling)**.** *Let* X *be a* T_4 *space, let* $\{F_1, \dots, F_n\}$ *be a finite family of closed subsets of* X *of order at most* n*, and let* $\{U_1, \dots, U_n\}$ *be open sets such that* $F_i \subset U_i$ *for* $i = 1, \dots, n$*. Then there exists a family* $\{V_1, \dots, V_n\}$ *of open subsets of* X *such that* $\{\overline{V_1}, \dots, \overline{V_n}\}$ *has order at most* n *and*

$$F_i \subset V_i \subset \overline{V_i} \subset U_i$$

for each $i = 1, \dots, n$*.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 55 (Characterization of covering dimension)**.** *For a* T_4 *space* X*, the following conditions are equivalent.*

- *(a)* $\dim X \leqslant n$*.*
- *(b) Every finite open cover of* X *has an open shrinking of order at most* n*.*
- *(c) Every finite open cover of* X *has a closed shrinking of order at most* n*.*
- *(d) Every finite open cover of* X *is refined by a finite closed cover of order at most* n*.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 56 (Sum theorem for the dimension dim)**.** *If a* T_4 *space* X *is the union of countably many closed subspaces* F_i *and* $\dim F_i \leqslant n$*, then* $\dim X \leqslant n$*.*

Důkaz. The proof was presented, it can be examined. ☐

Theorem 57. *If* X *is* T_4*, then* $\dim X \leqslant \operatorname{Ind} X$*.*

Důkaz. The proof was presented, it can be examined. ☐

5.1. Topological Dimension in Metrizable Spaces

Theorem 58. *If* X *is a metrizable space, then* $\dim X = \operatorname{Ind} X$*.*

Důkaz. The proof was presented only for the special case of compact X, this special case can be examined. ☐

the end of 12. lecture (7. 5. 2026)

Lemma 59. *Let X be a metrizable space and let $Z \subset X$ be strongly 0-dimensional. Then for every closed set $F \subset X$ and every open set $U \subset X$ with $F \subset U$ there exists an open set $V \subset X$ such that*

$$F \subset V \subset \overline{V} \subset U$$

and $Z \cap \partial V = \varnothing$.

Theorem 60. *Let X be a metrizable separable space. Then*

$$\operatorname{ind} X = \dim X = \operatorname{Ind} X.$$

Theorem 61. *Let X be a metrizable space and let $n \in \mathbb{N} \cup \{0\}$. Then the following statements are equivalent.*

(a) $\operatorname{Ind} X \leqslant n$,

(b) $X = Y \cup Z$, *where* $\operatorname{Ind} Y \leqslant n-1$ *and* $\operatorname{Ind} Z \leqslant 0$.

Corollary 62 (On separation)**.** *Let X be a metrizable space and let $n \in \mathbb{N} \cup \{0\}$. If $\operatorname{Ind} X \leqslant n$, then for every sequence of $(n+1)$ pairs of closed disjoint sets $(F_1, H_1), \ldots, (F_{n+1}, H_{n+1})$, there exist open sets U_i, $i = 1, \ldots, n+1$, such that*

$$F_i \subseteq U_i \subseteq \overline{U_i} \subseteq X \backslash H_i \quad \textit{and} \quad \bigcap_{i=1}^{n+1} \partial U_i = \varnothing.$$

Theorem 63. *Let X and Y be nonempty metrizable spaces. Then*

$$\operatorname{Ind}(X \times Y) \leqslant \operatorname{Ind} X + \operatorname{Ind} Y.$$

5.2. Dimension and Euclidean Spaces

Theorem 64 (Brouwer's fixed point theorem)**.** *Every continuous mapping $f\colon [0,1]^n \to [0,1]^n$ has a fixed point, i.e. there exists $x \in [0,1]^n$ such that $f(x) = x$.*

Theorem 65. *For every $n \in \mathbb{N}$ we have $\dim [0,1]^n = \dim \mathbb{R}^n = n$.*

Corollary 66. *If $n, m \in \mathbb{N}$ and $n \neq m$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.*