Lipschitz-free spaces: quantitative Schur property and weak* separability of the dual

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(joint work with L. Candido, B. Vejnar and with O. Kalenda)

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PART I

Gentle introduction to Lipschitz-free spaces

Motivation

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But, $JL_\infty \not\simeq c_0 \oplus JL_\infty/c_0$ (so Lipschitz selection cannot be linear).

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Will be needed: If $M \overset{\text{bi-Lip}}{\hookrightarrow} N$, then $\mathcal{F}(M) \hookrightarrow \mathcal{F}(N)$.

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- On the other hand, there is always Lipschitz selection (namely $\delta : X/Z \to \mathcal{F}(X/Z)$). Thus, if the linear selection does not exist, we have that

$$\mathcal{F}(X/Z) \stackrel{Lip}{\simeq} X/Z \oplus \ker \hat{Id}, \quad \text{but} \quad \mathcal{F}(X/Z) \not\simeq X/Z \oplus \ker \hat{Id}.$$

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Does $\mathcal{F}(X/Z) \overset{\hat{Id}}{\to} X/Z$ have linear selection?

- If yes, then $X \to X/Z$ has linear selection.
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Lifting property for separable spaces

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OPEN: Given two Lipschitz isomorphic SEPARABLE Banach spaces, are they linearly isomorphic?

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Proposition

Let $(M, \rho, 0)$ be a pointed p-metric space with $p \in (0,1]$. There exists only one (up to isometry) p-Banach space $\mathcal{F}_p(M)$ such that

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Albiac and Kalton (2009): There is a Banach space X such that $\mathcal{F}_p(X)$ is Lipschitz isomorphic but not linearly isomorphic to the p-Banach space $X \oplus \ker \hat{Id}$.

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- relation to metric geometry (e.g. M has property 1 iff $\mathcal{F}(M)$ has property 2 etc)

PART II

Weak* separability of the dual

Conjecture

- what we DO know about nonseparable Lipschitz-free spaces: $\ell_1(\operatorname{dens} M) \overset{c}{\hookrightarrow} \mathcal{F}(M)$
- what we DO NOT know: do they have Markushevich basis? Are they Plichko? Do they admit LUR renorming? ...

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Note: $\operatorname{dens} M \leq 2^\omega$ iff $|M| \leq 2^\omega$.

Note 2: WLOG M is complete.

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Characterization 2
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(Because the set $D = \{e_n \colon n \in \mathbb{N}\} \subset \ell_\infty$ separates the points of ℓ_1)

(Or because it is dual of the separable space ℓ_1)

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Example 3: $M \overset{bi-Lip}{\hookrightarrow} \ell_\infty \implies (\mathrm{Lip}_0(M), w^*)$ is separable
(Because then $\mathcal{F}(M) \hookrightarrow \mathcal{F}(\ell_\infty) \hookrightarrow \ell_\infty$)

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OPEN: Does ℓ_∞ bi-Lipschitz embed into $c_0(2^\omega)$?

PART IIa: CATEGORIAL ARGUMENTS ... THEOREM A

Step 1: Let X be a Banach space, then there is $\widehat{\mathrm{Id}} : \mathcal{F}(X) \to X$ with $\widehat{\mathrm{Id}} \circ \delta_X = \mathrm{Id}$.

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{Id}"] \arrow[d, hook, "\delta"] & X \\
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Step 2: Let $T : Y \to X$. Consider

$$PB := \{(\mu, y) \in \mathcal{F}(X) \times Y : \widehat{\mathrm{Id}}\mu = Ty\} \subset \mathcal{F}(X) \oplus_\infty Y.$$

Then $T \circ \pi_Y = \widehat{\mathrm{Id}} \circ \pi_{\mathcal{F}(X)}$.

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\widehat{\mathrm{Id}}"] & X \\ PB \arrow[u, dashed, "\pi_{\mathcal{F}(X)}"] \arrow[r, dashed, "\pi_Y"] & Y \arrow[u, "T"'] \end{tikzcd}$$

Categorial arguments

There exists (nonlinear) Lipschitz map $\varphi : Y \to PB$ with $\pi_Y \circ \varphi = \mathrm{Id}$.

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It is given by the formula $\varphi(y) := \big(\delta(Ty), y\big)$.

Theorem (Theorem A)

Let X be a Banach space and let there exist $T : \mathcal{F}(M) \to X$ which is one-to-one. Then there exists $S : \mathcal{F}(M) \to \mathcal{F}(X)$ which is one-to-one.

Proof: Apply the Step 2 above to $Y = \mathcal{F}(M)$. That is,

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In particular, $\varphi \circ \delta_M : M \to PB$ is Lipschitz, so there is $\widehat{\varphi \circ \delta_M} : \mathcal{F}(M) \to PB$ with $\widehat{\varphi \circ \delta_M} \circ \delta_M = \varphi \circ \delta_M$.

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It is one-to-one, because we have

$$\widehat{\mathrm{Id}} \circ S = T \circ \pi_{\mathcal{F}(M)} \circ \widehat{\varphi \circ \delta_M} = T$$

and T is one-to-one.

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  Hence, for $X := \mathcal{F}(C([0, \omega_1]))$ we have

$$(X^*, w^*) \text{ is separable but } (B_{X^*}, w^*) \text{ is not separable.}$$

Side remarks (provable using similar techniques)

Let X be a nonseparable WCG Banach space. Then

- There does not exist one-to-one $T : X \to \mathcal{F}(X)$

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Let X be a nonseparable WCG Banach space. Then

- There does not exist one-to-one $T : X \to \mathcal{F}(X)$
- $c_0 \hookrightarrow X \implies$ exists Banach Z such that

$$Z \overset{\text{bi-Lip}}{\simeq} X \quad \& \quad Z \text{ is not WCG}$$

(Z does not even have SCP).

PART IIb: TOPOLOGICAL ARGUMENTS ... THEOREMS B and C

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Crucial ingredience: Every metric space is paracompact (every open cover has locally finite open refinement).

For every $n \in \mathbb{N}$ there is locally finite open cover $\{U_n^s \colon s \in 2^\omega\}$ with $\operatorname{diam} U_n^s \leq \frac{1}{n}$.

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Countable enumeration: Let $\mathcal{B}$ be a countable basis of 2^ω and $\mathcal{N} := \{\bigcup_{i=1}^n B_i \colon B_1, \dots, B_n \in \mathcal{B}\}$.

The closed sets: are

$$F_{n,N} := \overline{\bigcup \{U_n^x \colon x \in N\}}, \qquad n \in \mathcal{N},\ N \in \mathcal{N}.$$

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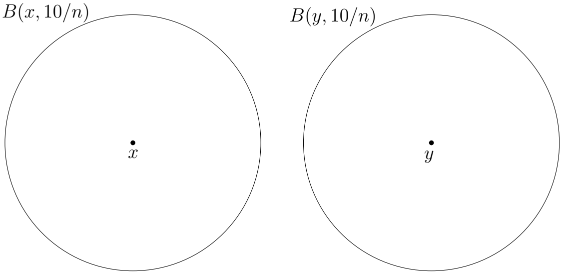

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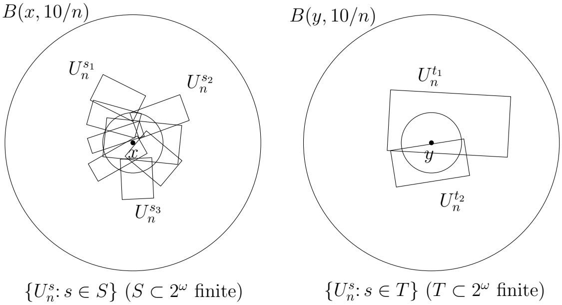

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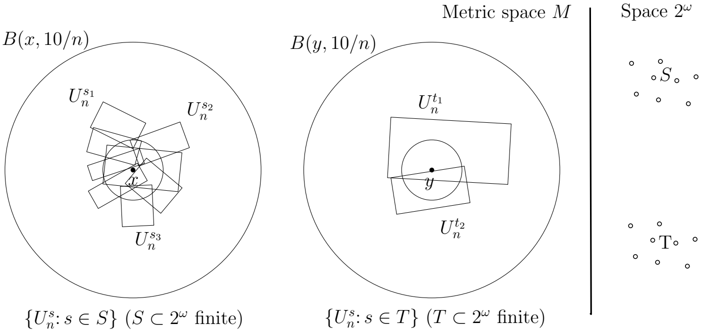

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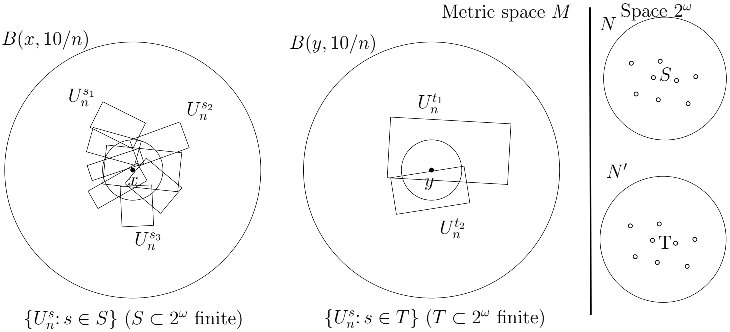

General strategy: Find a countable family of 1-Lipschitz functions (f_n) such that not only

$$M \ni x \overset{\varphi}{\mapsto} (f_n(x))_{n=1}^{\infty} \in \ell_\infty \text{ is one-to-one}$$

but also $\widehat{\varphi} : \mathcal{F}(M) \to \mathcal{F}(\ell_\infty)$ is one-to-one.

- How to find out whether $\mu \in \mathcal{F}(M)$ is zero?

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  Lemma: If M is complete, then $\mu \neq 0$ iff $\exists x \in M : \; x \in \operatorname{suppt} \mu$, that is

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- Find a condition such that $x \in \operatorname{suppt}(\mu) \implies \varphi(x) \in \operatorname{suppt} \widehat{\varphi}(\mu)$.

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General strategy: Find a countable family of 1-Lipschitz functions (f_n) such that not only

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but also $\widehat{\varphi} : \mathcal{F}(M) \to \mathcal{F}(\ell_\infty)$ is one-to-one.

- How to find out whether $\mu \in \mathcal{F}(M)$ is zero?
  Lemma: If M is complete, then $\mu \neq 0$ iff $\exists x \in M : \ x \in \operatorname{suppt} \mu$, that is
  $$\exists x \in M \, \forall \varepsilon > 0 \, \exists g \in \operatorname{Lip}_0(M) : \ \operatorname{suppt} g \subset B(x, \varepsilon) \ \& \ g(\mu) \neq 0.$$
- Find a condition such that $x \in \operatorname{suppt}(\mu) \implies \varphi(x) \in \operatorname{suppt} \widehat{\varphi}(\mu)$.

Proposition (Inspired by García-Lirola+Petitjean+Prochazka (2023))

Let M be a complete metric space with $\operatorname{dens} M \leq 2^\omega$. Suppose there exists a sequence of 1-Lipschitz functions $(f_n)_{n\in\mathbb{N}}$ such that

(*) *for every $x \in M$ there exists $\varepsilon > 0$ such that for every $g \in \operatorname{Lip}_0(M)$ with $\operatorname{suppt} g \subset B(x, \varepsilon)$ there exists $K > 0$ satisfying*

$$|g(y) - g(z)| \leq K \Big(\sup_{n\in\mathbb{N}} |f_n(y) - f_n(z)| \Big), \qquad y, z \in M.$$

Then $(\operatorname{Lip}_0(M), w^*)$ is separable.

Sufficient condition for $(\mathrm{Lip}_0(M), w^*)$ being separable:
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Proof: Consider for every $n \in \mathbb{N}$ and $i \in \{0, 1\}$

$$F_{n,i} := \{x_s \colon s \in 2^\omega \text{ such that } s(n) = i\}, \qquad f_{n,i} := d(\cdot, F_{n,i}).$$

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$$|g(y) - g(z)| = |g(x_s)| \leq \underbrace{\frac{|g(x_s)|}{\varepsilon}}_{=:K} d(x_s, F_{n,i}) \leq K|f_{n,i}(x_s) - f_{n,i}(x_t)|.$$

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Final Remarks: The conjecture is equivalent to:

- Under CH, to the fact that $(\mathrm{Lip}_0(\ell_\infty/c_0), w^*)$ is separable.

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Final Remarks: The conjecture is equivalent to:

- Under CH, to the fact that $(\mathrm{Lip}_0(\ell_\infty/c_0), w^*)$ is separable.
- To the fact that there is one-to-one $T : \mathcal{F}(M) \to X$, where

$$X = \left(\bigoplus_{2^\omega} \ell_\infty\right)_{\ell_1} \quad \text{or} \quad X = \left(\bigoplus_{2^\omega} \ell_\infty\right)_{c_0}$$

PART III

Quantitative Schur property

Schur property: If $x_n \xrightarrow{w} 0$, then $x_n \xrightarrow{\|\cdot\|} 0$.

c-**Schur property:** For any bounded (x_n) we have

$$\limsup_n \|x_n\| \leq c \sup\{\limsup_n |x^*(x_n)| : \ x^* \in B_{X^*}\}.$$

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Rosenthal's ℓ_1 theorem implies: X is Schur iff any uniformly separated bounded sequence has a subsequence equivalent to ℓ_1-basis.

c-strong Schur property: For any $\delta \in (0, 2]$ and any $\varepsilon > 0$ any normalized δ-separated sequence in X admits a subsequence $(\frac{2c}{\delta} + \varepsilon)$-equivalent to the standard basis of ℓ_1.

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$$c\text{-Schur} \implies c\text{-strong Schur} \implies (2c+1)\text{-Schur}$$

Aliaga, Gartland, Petitjean, Procházka, 2022: M complete, then

$$\mathcal{F}(M) \text{ Schur } \Leftrightarrow M \text{ is p1u}$$

(M is p1u = no bi-Lipschitz copy of compact $K \subset \mathbb{R}$ with $\lambda(K) > 0$.)

Moreover, if M is proper and p1u, then $\mathcal{F}(M)$ is 1-Schur (because its predual $\mathrm{lip}_0(M)$ almost isometrically embeds into c_0).

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- WLOG distances are integers

  Proof: Because $(x, y) \mapsto \lceil cd(x, y)\rceil$ is $(1 + \varepsilon)$-equivalent metric for big enough c.

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(And similarly $d(x_1, y_1) \leq 3d(x_m, y_n)$.)

Reformulation (almost): Assume $\|\mu_n\| \to \alpha > 0$.
Does there exist $g \in cB_{\mathrm{Lip}_0(M)}$ such that $g(\mu_{n_k}) \geq \|\mu_{n_k}\|$, $k \in \mathbb{N}$?

Lemma: For any $\mu \in \mathcal{F}(M)$ we have

$$\|\mu\| = \max\{f(\mu)\colon f \in B_{\mathrm{Lip}_0(M)} \text{ and } f(M) \subset \mathbb{Z}\}$$

Example: Assume $\mu_n = \alpha_n \delta(x_n) - \beta_n \delta(y_n)$, $n \in \mathbb{N}$. WLOG:

$$d(x_n, y_n) \equiv const, \quad d(y_n, 0) \equiv const, \quad d(x_n, 0) \equiv const, \quad d(x_n, x_m) \equiv const, \qquad n$$

$$d(x_n, y_m) \equiv const, \quad d(x_m, y_n) \equiv const, \qquad m < n.$$

Note that for $m < n$ we have

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Find a function $f_n \in B_{\mathrm{Lip}_0(\{0, x_n, y_n\})}$ with $\mathrm{Rng}\, f_n \subset \mathbb{Z}$ with $f_n(\mu_n) = \|\mu_n\|$.

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Consider $g(x) := f_n(x)$, $x \in \mathrm{Rng}\, f_n$. Then $g(\mu_n) = \|\mu_n\|$, $n \in \mathbb{N}$ and g is 3-Lipschitz:

$$|g(x_n) - g(y_m)| = |f_1(x_1) - f_1(y_1)| \leq d(x_1, y_1) \leq 3d(x_n, y_m), \quad m < n$$

(The other cases are similar or easier to handle ..)

Theorem: M is uniformly discrete, then $\mathcal{F}(M)$ is 3-Schur.

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Theorem: M is ultrametric, then $\mathcal{F}(M)$ is 1-Schur.

Thank you for your attention!