Descriptive set theory in Banach spaces III

Marek Cúth

Summer School 2026: Topics in Banach Space Theory
Castro Urdiales

PART FIVE: Complexity of the isometry class of some $C(K)$ spaces - upper bound

Cantor set

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Let X be a Banach space which is L_1 predual (G_δ condition). There is $F_{\sigma,\delta}$ condition

$$\forall \varepsilon > 0 : \ \overline{\text{aco}}\Big\{ a \in S_X : \ \forall y \in S_X : \ \max\{\|a-y\|, \|a+y\|\} \geq 2-\varepsilon \Big\} = B_X. \tag{1}$$

which is equivalent to the fact that $X \equiv C(K)$ with K zero-dimensional.

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Let X be a Banach space which is L_1 predual (G_δ condition). There is $F_{\sigma,\delta}$ condition

$$\forall \varepsilon > 0 : \ \overline{\mathrm{aco}}\Big\{a \in S_X : \ \forall y \in S_X : \ \max\{\|a-y\|, \|a+y\|\} \geq 2-\varepsilon\Big\} = B_X. \tag{1}$$

which is equivalent to the fact that $X \equiv C(K)$ with K zero-dimensional.

Moreover, $\{\mu \in \mathcal{B};\ X_\mu \text{ has Daugavet property}\}$ is G_δ.
(*López-Pérez, Martínez Vañó, Rueda Zoca 26*).

Well-known:

- $C(K)$ space has Daugavet property iff K does not have isolated points,
- K zero-dimensional+metric does not have isolated points if and only if $K \approx 2^{\mathbb{N}}$.

Cantor set

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Let X be a Banach space which is L_1 predual (G_δ condition). There is $F_{\sigma,\delta}$ condition

$$\forall \varepsilon > 0 : \ \overline{\text{aco}}\Big\{a \in S_X : \ \forall y \in S_X : \ \max\{\|a-y\|, \|a+y\|\} \geq 2-\varepsilon\Big\} = B_X. \tag{1}$$

which is equivalent to the fact that $X \equiv C(K)$ with K zero-dimensional.

Moreover, $\{\mu \in \mathcal{B};\ X_\mu$ has Daugavet property$\}$ is G_δ.
(*López-Pérez, Martínez Vañó, Rueda Zoca 26*).

Well-known:

- $C(K)$ space has Daugavet property iff K does not have isolated points,
- K zero-dimensional+metric does not have isolated points if and only if $K \approx 2^{\mathbb{N}}$.

Corollary: $C(2^{\mathbb{N}})$ is up to isometry the only sep ∞-dim Banach space which is L_1-predual, satisfies (1) and has Daugavet property.

Recall: $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Let X be a Banach space which is L_1 predual (G_δ condition). There is $F_{\sigma,\delta}$ condition

$$\forall\varepsilon>0:\ \overline{\text{aco}}\Big\{a\in S_X:\ \forall y\in S_X:\ \max\{\|a-y\|,\|a+y\|\}\geq 2-\varepsilon\Big\}=B_X. \tag{1}$$

which is equivalent to the fact that $X\equiv C(K)$ with K zero-dimensional.

Moreover, $\{\mu\in\mathcal{B};\ X_\mu$ has Daugavet property$\}$ is G_δ.
(*López-Pérez, Martínez Vañó, Rueda Zoca 26*).

Well-known:

- $C(K)$ space has Daugavet property iff K does not have isolated points,
- K zero-dimensional+metric does not have isolated points if and only if $K\approx 2^{\mathbb{N}}$.

Corollary: $C(2^{\mathbb{N}})$ is up to isometry the only sep ∞-dim Banach space which is L_1-predual, satisfies (1) and has Daugavet property.

Theorem

$\langle C(2^{\mathbb{N}})\rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Cantor set

Recall: $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Let X be a Banach space which is L_1 predual (G_δ condition). There is $F_{\sigma,\delta}$ condition

$$\forall\varepsilon>0:\ \overline{\text{aco}}\Big\{a\in S_X:\ \forall y\in S_X:\ \max\{\|a-y\|,\|a+y\|\}\geq 2-\varepsilon\Big\}=B_X. \tag{1}$$

which is equivalent to the fact that $X\equiv C(K)$ with K zero-dimensional.

Moreover, $\{\mu\in\mathcal{B};\ X_\mu$ has Daugavet property$\}$ is G_δ.
(*López-Pérez, Martínez Vañó, Rueda Zoca 26*).

Well-known:

- $C(K)$ space has Daugavet property iff K does not have isolated points,
- K zero-dimensional+metric does not have isolated points if and only if $K\approx 2^{\mathbb{N}}$.

Corollary: $C(2^{\mathbb{N}})$ is up to isometry the only sep ∞-dim Banach space which is L_1-predual, satisfies (1) and has Daugavet property.

Theorem

$\langle C(2^{\mathbb{N}})\rangle_{\equiv}$ *is* $F_{\sigma,\delta}$.

For isometry classes of $C(K)$ spaces with K countable due to (1), it suffices to find characterization among $C(K)$ spaces.

Cantor set

Recall: $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$.

Let X be a Banach space which is L_1 predual (G_δ condition). There is $F_{\sigma,\delta}$ condition

$$\forall\varepsilon>0:\ \overline{\text{aco}}\Big\{a\in S_X:\ \forall y\in S_X:\ \max\{\|a-y\|,\|a+y\|\}\geq 2-\varepsilon\Big\}=B_X. \tag{1}$$

which is equivalent to the fact that $X\equiv C(K)$ with K zero-dimensional.

Moreover, $\{\mu\in\mathcal{B};\ X_\mu$ has Daugavet property$\}$ is G_δ.
(*López-Pérez, Martínez Vañó, Rueda Zoca 26*).

Well-known:

- $C(K)$ space has Daugavet property iff K does not have isolated points,
- K zero-dimensional+metric does not have isolated points if and only if $K\approx 2^{\mathbb{N}}$.

Corollary: $C(2^{\mathbb{N}})$ is up to isometry the only sep ∞-dim Banach space which is L_1-predual, satisfies (1) and has Daugavet property.

Theorem

$\langle C(2^{\mathbb{N}})\rangle_{\equiv}$ is $F_{\sigma,\delta}$.

For isometry classes of $C(K)$ spaces with K countable due to (1), it suffices to find characterization among $C(K)$ spaces. We will use Szlenk index .. we should recall some notation ..

Cantor-Bendixson derivative $K^{(\alpha)}$:

- $K^{(0)} = K$
- $K' = K^{(1)} = \{x \in K : \ x \text{ is an accumulation point of } K\}$
- $K^{(\alpha+1)} = (K^{(\alpha)})^{(1)}$
- $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$, if α is a limit ordinal.

Mazurkiewicz-Sierpinski theorem:

$$K \text{ countable } \Rightarrow \exists!\alpha < \omega_1 \ \exists! n \in \mathbb{N} : K \approx [0, \omega^\alpha \cdot n] \text{ homeomorphically}$$

(α is the smallest ordinal with $K^{(\alpha)}$ finite, $n = |K^{(\alpha)}|$)

Hence, we need to find characterization of $C([0, \omega^\alpha \cdot n])$.

Szlenk derivative: Let $K \subset X^*$ be w^*-compact and let $\varepsilon > 0$.

$$s_\varepsilon(K) = \left\{x \in K : \ \operatorname{diam}(V \cap K) \geq \varepsilon \text{ for every } w^*\text{-neighbourhood } V \ni x\right\}.$$

The transfinite derivatives are defined recursively by

$$s_\varepsilon^0(K) = K, \quad s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha(K)), \quad s_\varepsilon^\alpha(K) = \bigcap_{\beta<\alpha} s_\varepsilon^\beta(K) \text{for } \alpha \text{ limit.}$$

Cantor-Bendixson derivative $K^{(\alpha)}$:

- $K^{(0)} = K$
- $K' = K^{(1)} = \{x \in K : \ x \text{ is an accumulation point of } K\}$
- $K^{(\alpha+1)} = (K^{(\alpha)})^{(1)}$
- $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$, if α is a limit ordinal.

Mazurkiewicz-Sierpinski theorem:

$$K \text{ countable } \Rightarrow \exists! \alpha < \omega_1 \ \exists! n \in \mathbb{N} : K \approx [0, \omega^\alpha \cdot n] \text{ homeomorphically}$$

(α is the smallest ordinal with $K^{(\alpha)}$ finite, $n = |K^{(\alpha)}|$)

Hence, we need to find characterization of $C([0, \omega^\alpha \cdot n])$.

Szlenk derivative: Let $K \subset X^*$ be w^*-compact and let $\varepsilon > 0$.

$$s_\varepsilon(K) = \left\{x \in K : \ \operatorname{diam}(V \cap K) \geq \varepsilon \text{ for every } w^*\text{-neighbourhood } V \ni x\right\}.$$

The transfinite derivatives are defined recursively by

$$s_\varepsilon^0(K) = K, \quad s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha(K)), \quad s_\varepsilon^\alpha(K) = \bigcap_{\beta<\alpha} s_\varepsilon^\beta(K) \text{for } \alpha \text{ limit.}$$

Characterization of $C([0,\omega^\alpha \cdot k])$

The isometric characterization we shall use is the following.

Theorem

Let X be a separable Banach space, $\alpha < \omega_1$ countable nonzero ordinal and $k \in \mathbb{N}$. Then X is isometric to $\mathcal{C}(\omega^\alpha \cdot k)$ if and only if the following conditions hold.

- *X is isometric to some $\mathcal{C}(K)$ space with K zero-dimensional ($F_{\sigma,\delta}$ condition mentioned above),*
- *$s_2^\alpha(B_{X^*})$ is a nonempty subset of k-dimensional space, and if $k > 1$ then it is not a subset of $(k-1)$-dimensional space.*

Borel hierarchy:

For a countable ordinal $\alpha < \omega_1$:

- Σ_1^0: open sets, Π_1^0: closed sets.
- For $\alpha > 1$,

$$\Sigma_\alpha^0 = \left\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \in \bigcup_{\beta<\alpha} \Pi_\beta^0 \right\}, \quad \Pi_\alpha^0 = \left\{ \bigcap_{n\in\mathbb{N}} A_n : A_n \in \bigcup_{\beta<\alpha} \Sigma_\beta^0 \right\}.$$

(G_δ sets are $\mathbf{\Pi}_2^0$, F_σ are $\mathbf{\Sigma}_2^0$ etc.)

Given a class $\mathbf{\Gamma}$ of subsets of Polish spaces, we say that a subset A of a Polish space X is a $D_2(\mathbf{\Gamma})$ set if $A = B \setminus C$ for some $B, C \in \mathbf{\Gamma}(X)$

Characterization of $C([0, \omega^\alpha \cdot k])$

The isometric characterization we shall use is the following.

Theorem

Let X be a separable Banach space, $\alpha < \omega_1$ countable nonzero ordinal and $k \in \mathbb{N}$. Then X is isometric to $\mathcal{C}(\omega^\alpha \cdot k)$ if and only if the following conditions hold.

- *X is isometric to some $\mathcal{C}(K)$ space with K zero-dimensional ($F_{\sigma,\delta}$ condition mentioned above),*
- *$s_2^\alpha(B_{X^*})$ is a nonempty subset of k-dimensional space, and if $k > 1$ then it is not a subset of $(k-1)$-dimensional space.*

Borel hierarchy:

For a countable ordinal $\alpha < \omega_1$:

- Σ_1^0: open sets, Π_1^0: closed sets.
- For $\alpha > 1$,

$$\Sigma_\alpha^0 = \left\{ \bigcup_{n \in \mathbb{N}} A_n : A_n \in \bigcup_{\beta < \alpha} \Pi_\beta^0 \right\}, \quad \Pi_\alpha^0 = \left\{ \bigcap_{n \in \mathbb{N}} A_n : A_n \in \bigcup_{\beta < \alpha} \Sigma_\beta^0 \right\}.$$

(G_δ sets are $\mathbf{\Pi}_2^0$, F_σ are $\mathbf{\Sigma}_2^0$ etc.)

Given a class $\mathbf{\Gamma}$ of subsets of Polish spaces, we say that a subset A of a Polish space X is a $D_2(\mathbf{\Gamma})$ set if $A = B \setminus C$ for some $B, C \in \mathbf{\Gamma}(X)$

Complexity of $C(K)$ and K

Summary of the main result concerning the upper bounds.

Theorem

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$ set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$ set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$ set.*

Using "continuous reduction $K \mapsto C(K)$" we obtain also the following.

Theorem

Let X be an uncountable metrizable compact space, β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The homeomorphism class of $[0, \omega^{\beta+n}]$ in $\mathcal{K}(X)$ is $\mathbf{\Pi}^0_{\beta+2n+1}$.*
- *The homeomorphism class of $[0, \omega^{\beta+n} \cdot k]$ in $\mathcal{K}(X)$ is $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$ for every $k \in \mathbb{N}$ with $k \geq 2$.*

We can also obtain that homeomorphism class of $2^{\mathbb{N}}$ in $\mathcal{K}(X)$ is $F_{\sigma,\delta}$, but it follows from known results that it is even G_δ-complete.

Summary of the main result concerning the upper bounds.

Theorem

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$ set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$ set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$ set.*

Using "continuous reduction $K \mapsto C(K)$" we obtain also the following.

Theorem

Let X be an uncountable metrizable compact space, β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The homeomorphism class of $[0, \omega^{\beta+n}]$ in $\mathcal{K}(X)$ is $\mathbf{\Pi}^0_{\beta+2n+1}$.*
- *The homeomorphism class of $[0, \omega^{\beta+n} \cdot k]$ in $\mathcal{K}(X)$ is $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$ for every $k \in \mathbb{N}$ with $k \geq 2$.*

We can also obtain that homeomorphism class of $2^{\mathbb{N}}$ in $\mathcal{K}(X)$ is $F_{\sigma,\delta}$, but it follows from known results that it is even G_δ-complete.

Complexity of $C(K)$ and K

Summary of the main result concerning the upper bounds.

Theorem

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$ set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$ set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$ set.*

Using "continuous reduction $K \mapsto C(K)$" we obtain also the following.

Theorem

Let X be an uncountable metrizable compact space, β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The homeomorphism class of $[0, \omega^{\beta+n}]$ in $\mathcal{K}(X)$ is $\mathbf{\Pi}^0_{\beta+2n+1}$.*
- *The homeomorphism class of $[0, \omega^{\beta+n} \cdot k]$ in $\mathcal{K}(X)$ is $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$ for every $k \in \mathbb{N}$ with $k \geq 2$.*

We can also obtain that homeomorphism class of $2^{\mathbb{N}}$ in $\mathcal{K}(X)$ is $F_{\sigma,\delta}$, but it follows from known results that it is even G_δ-complete.

PART SIX: Complexity of the isometry class of c - lower bound

Recall: $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from* A *to* B, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is Γ-hard if $A \leq_W B$ whenever $A \in \Gamma(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$. Then

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c\rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from A to B*, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is Γ-hard if $A \leq_W B$ whenever $A \in \Gamma(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} :\ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 :\ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} :\ a_{n,m} = 1\}$. Then

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} :\ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c\rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from A to B*, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is $\mathbf{\Gamma}$-hard if $A \leq_W B$ whenever $A \in \mathbf{\Gamma}(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c\rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} :\ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 :\ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} :\ a_{n,m} = 1\}$. Then

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} :\ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c\rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from A to B*, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is $\mathbf{\Gamma}$-hard if $A \leq_W B$ whenever $A \in \mathbf{\Gamma}(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j \in \mathbb{N}} \in 2^{\mathbb{N} \times \mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$. Then

  $$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c \rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from A to B*, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is $\mathbf{\Gamma}$-hard if $A \leq_W B$ whenever $A \in \mathbf{\Gamma}(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} : \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j \in \mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : a_{n,m} = 1\}$. Then

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \forall n \in \mathbb{N}\; A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c \rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from* A *to* B, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is $\mathbf{\Gamma}$-hard if $A \leq_W B$ whenever $A \in \mathbf{\Gamma}(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j \in \mathbb{N}} \in 2^{\mathbb{N} \times \mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$. Then

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c \rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c \rangle_\equiv$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from A to B*, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is $\mathbf{\Gamma}$-hard if $A \leq_W B$ whenever $A \in \mathbf{\Gamma}(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c \rangle_\equiv$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j \in \mathbb{N}} \in 2^{\mathbb{N} \times \mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$. Then

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c \rangle_\equiv$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

Recall: $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$. What about the lower bound?

Method of obtaining the lower bound:

Given subsets $A \subseteq X$ and $B \subseteq Y$ of Polish spaces X and Y, a continuous function $f : X \to Y$ such that $A = f^{-1}(B)$ is *continuous reduction from A to B*, in this case we write $A \leq_W B$.

We say that a set $B \subseteq Y$ is $\mathbf{\Gamma}$-hard if $A \leq_W B$ whenever $A \in \mathbf{\Gamma}(X)$ and X is Polish and zero-dimensional.

Important facts:

- If A is $F_{\sigma,\delta}$-hard, then it is not $G_{\delta,\sigma}$. We want to prove that $\langle c \rangle_{\equiv}$ is $F_{\sigma,\delta}$-hard.
- If $A \leq_W B$ and A is $F_{\sigma,\delta}$-hard, then B is also $F_{\sigma,\delta}$-hard.
- Essentially known example of a $F_{\sigma,\delta}$-hard set is the following. Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). Given matrix $A = (a_{i,j})_{i,j \in \mathbb{N}} \in 2^{\mathbb{N} \times \mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$. Then

  $$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ A_n \notin U\}$$

  is $F_{\sigma,\delta}$-hard (matrices, where on every row we have finitely many 1's).

So we need to prove that $P_3 \leq_W \langle c \rangle_{\equiv}$, that is, a continuous construction, where we assign infinite-dimensional Banach space X_A to a matrix A in such a way that $X_A \equiv c_0$ iff $A \in P_3$.

The construction: Put $T := \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$.

The construction: Put $T := \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$.

Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$.

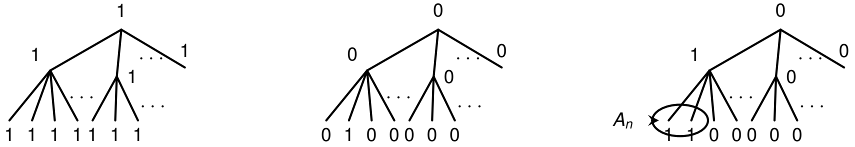
The construction: Put $T := \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$.

Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} :\ a_{n,m} = 1\}$.

Put

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

and consider $X(A) := \overline{\mathrm{span}}\{\chi_S;\ S \in E(A)\} \subset \ell_\infty(T)$.



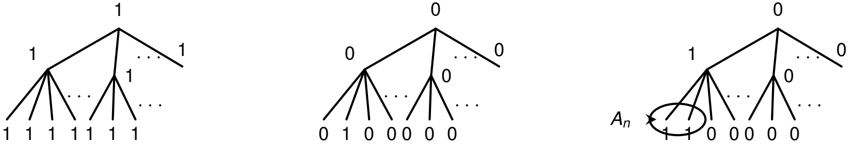
The construction: Put $T := \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$.

Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} :\ a_{n,m} = 1\}$.

Put

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

and consider $X(A) := \overline{\operatorname{span}}\{\chi_S;\ S \in E(A)\} \subset \ell_\infty(T)$.



Define $\mu_A \in c_{00}(T)$, for $F \subset \mathbb{N}$ and $G \subset \mathbb{N}^2$ finite as

$$\mu_A\Big(\lambda_\emptyset e_\emptyset + \sum_{n\in F} \lambda_n e_n + \sum_{(n,m)\in G} \lambda_{(n,m)} e_{(n,m)}\Big) := \Big\|\lambda_\emptyset \chi_T + \sum_{(n,m)\in G} \lambda_{(n,m)} \chi_{(n,m)} + \sum_{n\in F} \lambda_n \chi_{\{n\}\cup(\{n\}\times A_n)}\Big\|,$$

Then $X(A) \equiv X_{\mu(A)}$

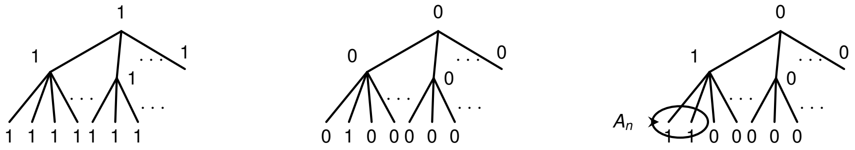
The construction: Put $T := \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$.

Given matrix $A = (a_{i,j})_{i,j\in\mathbb{N}} \in 2^{\mathbb{N}\times\mathbb{N}}$, denote $A_n := \{m \in \mathbb{N} : \ a_{n,m} = 1\}$.

Put

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

and consider $X(A) := \overline{\operatorname{span}}\{\chi_S;\ S \in E(A)\} \subset \ell_\infty(T)$.



Define $\mu_A \in c_{00}(T)$, for $F \subset \mathbb{N}$ and $G \subset \mathbb{N}^2$ finite as

$$\mu_A\Big(\lambda_\emptyset e_\emptyset + \sum_{n\in F} \lambda_n e_n + \sum_{(n,m)\in G} \lambda_{(n,m)} e_{(n,m)}\Big) := \Big\|\lambda_\emptyset \chi_T + \sum_{(n,m)\in G} \lambda_{(n,m)} \chi_{(n,m)} + \sum_{n\in F} \lambda_n \chi_{\{n\}\cup(\{n\}\times A_n)}\Big\|,$$

Then $X(A) \equiv X_{\mu(A)}$ and $2^{\mathbb{N}\times\mathbb{N}} \ni A \mapsto \mu(A) \in \mathcal{B}$ is continuous, because enlarging F and G we may assume $n \in F$ if and only if $(n,m) \in G$ for some $m \in \mathbb{N}$, and then the above is equal to

$$\max\Big\{|\lambda_\emptyset|, \max_{n\in F} |\lambda_\emptyset + \lambda_n|, \max_{(n,m)\in G} |\lambda_\emptyset + \lambda_{(n,m)} + \lambda_n a_{n,m}|\Big\}.$$

Reduction - in language of Boolean algebras

Stone duality: Given a set T and Boolean algebra $B \subset \mathcal{P}(T)$, there exists a unique (up to homeomorphism) zero-dimensional compact space K_B for which there exists an isomorphism of Boolean algebras $s : B \to \mathrm{Clop}(K_B)$.

Reduction - in language of Boolean algebras

Stone duality: Given a set T and Boolean algebra $B \subset \mathcal{P}(T)$, there exists a unique (up to homeomorphism) zero-dimensional compact space K_B for which there exists an isomorphism of Boolean algebras $s : B \to \mathrm{Clop}(K_B)$. Then $C(K_B)$ is linearly isometric to

$$\overline{\mathrm{span}}\{\chi_S;\ S \in B\} \subset \ell_\infty(T).$$

Reduction - in language of Boolean algebras

Stone duality: Given a set T and Boolean algebra $B \subset \mathcal{P}(T)$, there exists a unique (up to homeomorphism) zero-dimensional compact space K_B for which there exists an isomorphism of Boolean algebras $s : B \to \mathrm{Clop}(K_B)$. Then $C(K_B)$ is linearly isometric to

$$\overline{\mathrm{span}}\{\chi_S;\ S \in B\} \subset \ell_\infty(T).$$

Observation 1: If the Boolean algebra B is generated by $E \subset \mathcal{P}(T)$ with $T \in E$ and $E \cup \{0\}$ is closed under finite intersections, then

$$C(K_B) \equiv \overline{\mathrm{span}}\{\chi_S;\ S \in E\} \subset \ell_\infty(T).$$

Reduction - in language of Boolean algebras

Stone duality: Given a set T and Boolean algebra $B \subset \mathcal{P}(T)$, there exists a unique (up to homeomorphism) zero-dimensional compact space K_B for which there exists an isomorphism of Boolean algebras $s : B \to \mathrm{Clop}(K_B)$. Then $C(K_B)$ is linearly isometric to

$$\overline{\mathrm{span}}\{\chi_S;\ S \in B\} \subset \ell_\infty(T).$$

Observation 1: If the Boolean algebra B is generated by $E \subset \mathcal{P}(T)$ with $T \in E$ and $E \cup \{0\}$ is closed under finite intersections, then

$$C(K_B) \equiv \overline{\mathrm{span}}\{\chi_S;\ S \in E\} \subset \ell_\infty(T).$$

In our construction,

$$E(A) = \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

$$X(A) = \overline{\mathrm{span}}\{\chi_S;\ S \in E(A)\} \subset \ell_\infty(T),$$

so $X(A) \equiv C(K_A)$, where K_A is the Stone space corresponding to the Boolean algebra B_A generated by $E(A)$.

Reduction - in language of Boolean algebras

Stone duality: Given a set T and Boolean algebra $B \subset \mathcal{P}(T)$, there exists a unique (up to homeomorphism) zero-dimensional compact space K_B for which there exists an isomorphism of Boolean algebras $s : B \to \mathrm{Clop}(K_B)$. Then $C(K_B)$ is linearly isometric to

$$\overline{\mathrm{span}}\{\chi_S;\ S \in B\} \subset \ell_\infty(T).$$

Observation 1: If the Boolean algebra B is generated by $E \subset \mathcal{P}(T)$ with $T \in E$ and $E \cup \{0\}$ is closed under finite intersections, then

$$C(K_B) \equiv \overline{\mathrm{span}}\{\chi_S;\ S \in E\} \subset \ell_\infty(T).$$

In our construction,

$$E(A) = \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

$$X(A) = \overline{\mathrm{span}}\{\chi_S;\ S \in E(A)\} \subset \ell_\infty(T),$$

so $X(A) \equiv C(K_A)$, where K_A is the Stone space corresponding to the Boolean algebra B_A generated by $E(A)$.

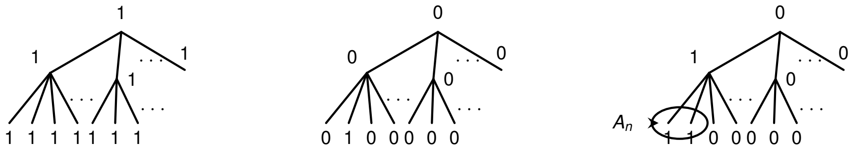
For which matrices A is it true that K_A has only one non-isolated point?

Non-isolated points: The set $(K_A)'$ of non-isolated point of K_A is homeomorphic to the Stone space $K_{B'}$, where $B' = B_A/I(\mathrm{At}\, B_A)$ and $I(\mathrm{At}\, B_A)$ is the ideal generated by atoms of B_A.

Recall:

$$E(A) := \{T\} \cup \left\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\right\} \cup \left\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\right\} \subset \mathcal{P}(T)$$

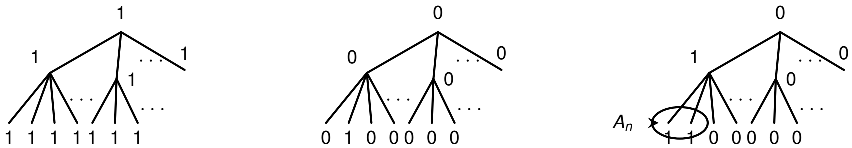
We need to find atoms of Boolean algebra generated by $E(A)$.



Recall:

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

We need to find atoms of Boolean algebra generated by $E(A)$.



We have

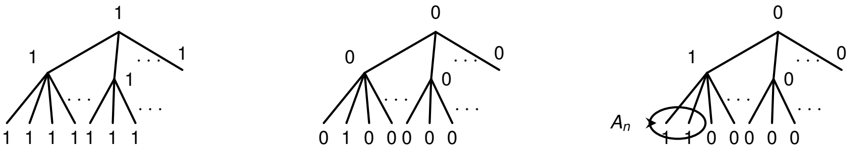
$$\mathrm{At}(\mathrm{B_A}) = \Big\{\{(n,m)\}:\ (n,m) \in \mathbb{N} \times \mathbb{N}\Big\} \cup \Big\{\{n\}:\ n \in \mathbb{N} \text{ is such that } A_n \text{ is finite}\Big\}.$$

Atoms of K_A

Recall:

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

We need to find atoms of Boolean algebra generated by $E(A)$.



We have

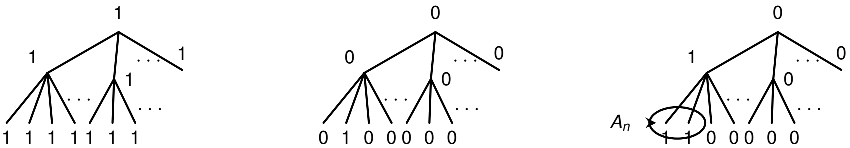
$$\mathrm{At}(\mathrm{B_A}) = \Big\{\{(n,m)\}:\ (n,m) \in \mathbb{N} \times \mathbb{N}\Big\} \cup \Big\{\{n\}:\ n \in \mathbb{N} \text{ is such that } A_n \text{ is finite}\Big\}.$$

Non-isolated points: The set $(K_A)'$ of non-isolated point of K_A is homeomorphic to the Stone space $K_{B'}$, where $B' = B_A / I(\mathrm{At}\, B_A)$ and $I(\mathrm{At}\, B_A)$ is the ideal generated by atoms of B_A.

Recall:

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

We need to find atoms of Boolean algebra generated by $E(A)$.



We have

$$\mathrm{At}(\mathrm{B_A}) = \Big\{\{(n,m)\}:\ (n,m) \in \mathbb{N} \times \mathbb{N}\Big\} \cup \Big\{\{n\}:\ n \in \mathbb{N} \text{ is such that } A_n \text{ is finite}\Big\}.$$

Non-isolated points: The set $(K_A)'$ of non-isolated point of K_A is homeomorphic to the Stone space $K_{B'}$, where $B' = B_A/I(\mathrm{At}\, B_A)$ and $I(\mathrm{At}\, B_A)$ is the ideal generated by atoms of B_A.

Hence,

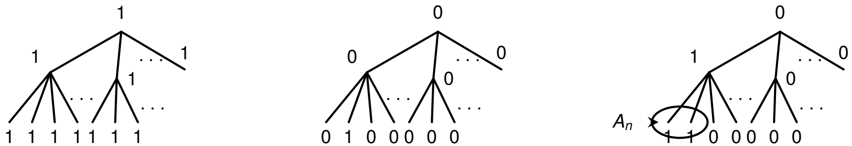
- $(K_A)'$ consists of just one point iff A_n is finite for every $n \in \mathbb{N}$.

Atoms of K_A

Recall:

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

We need to find atoms of Boolean algebra generated by $E(A)$.



We have

$$\mathrm{At}(\mathrm{B_A}) = \Big\{\{(n,m)\}:\ (n,m) \in \mathbb{N} \times \mathbb{N}\Big\} \cup \Big\{\{n\}:\ n \in \mathbb{N} \text{ is such that } A_n \text{ is finite}\Big\}.$$

Non-isolated points: The set $(K_A)'$ of non-isolated point of K_A is homeomorphic to the Stone space $K_{B'}$, where $B' = B_A / I(\mathrm{At}\, B_A)$ and $I(\mathrm{At}\, B_A)$ is the ideal generated by atoms of B_A.

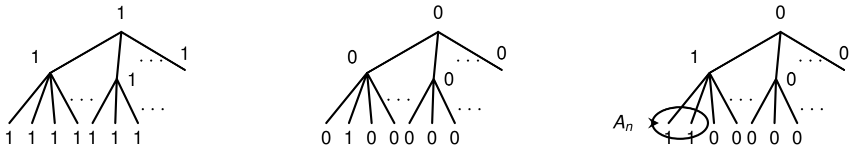
Hence,

- $(K_A)'$ consists of just one point iff A_n is finite for every $n \in \mathbb{N}$.
- $|(K_A)'| = k + 1$ iff $|\{n;\ A_n \text{ is infinite }\}| = k$.

Recall:

$$E(A) := \{T\} \cup \Big\{\{(n,m)\};\ (n,m) \in \mathbb{N}^2\Big\} \cup \Big\{\{n\} \cup (\{n\} \times A_n);\ n \in \mathbb{N}\Big\} \subset \mathcal{P}(T)$$

We need to find atoms of Boolean algebra generated by $E(A)$.



We have

$$\mathrm{At}(\mathrm{B_A}) = \Big\{\{(n,m)\}:\ (n,m) \in \mathbb{N} \times \mathbb{N}\Big\} \cup \Big\{\{n\}:\ n \in \mathbb{N} \text{ is such that } A_n \text{ is finite}\Big\}.$$

Non-isolated points: The set $(K_A)'$ of non-isolated point of K_A is homeomorphic to the Stone space $K_{B'}$, where $B' = B_A / I(\mathrm{At}\, B_A)$ and $I(\mathrm{At}\, B_A)$ is the ideal generated by atoms of B_A.

Hence,

- $(K_A)'$ consists of just one point iff A_n is finite for every $n \in \mathbb{N}$.
- $|(K_A)'| = k + 1$ iff $|\{n;\ A_n \text{ is infinite }\}| = k$.
- if $(K_A)'$ is infinite, then $K_A \approx [0, \omega^2]$.

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N} \ \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),
- $C([0, \omega \cdot k]) \Leftrightarrow |\{n;\ A_n \in U\}| = k - 1$,

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),
- $C([0, \omega \cdot k]) \Leftrightarrow |\{n;\ A_n \in U\}| = k - 1$,
- $C([0, \omega^2]) \Leftrightarrow |\{n;\ A_n \notin U\}| = \omega$.

Hence, we need to estimate complexity of $\{A \in 2^{\mathbb{N}\times\mathbb{N}};\ |\{n;\ A_n \notin U\}| = k\}$ for $k = 0$ (and maybe even for $k \in \mathbb{N}$).

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),
- $C([0,\omega\cdot k]) \Leftrightarrow |\{n;\ A_n \in U\}| = k-1$,
- $C([0,\omega^2]) \Leftrightarrow |\{n;\ A_n \notin U\}| = \omega$.

Hence, we need to estimate complexity of $\{A \in 2^{\mathbb{N}\times\mathbb{N}};\ |\{n;\ A_n \notin U\}| = k\}$ for $k = 0$ (and maybe even for $k \in \mathbb{N}$).

Proposition

Let X be a zero-dimensional Polish space, $\alpha \geq 1$ a countable ordinal and $k \in \mathbb{N}$. If $U \subseteq X$ is $\mathbf{\Pi}^0_\alpha$-complete set, then the following holds.

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} :\ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 :\ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),
- $C([0, \omega \cdot k]) \Leftrightarrow |\{n;\ A_n \in U\}| = k - 1$,
- $C([0, \omega^2]) \Leftrightarrow |\{n;\ A_n \notin U\}| = \omega$.

Hence, we need to estimate complexity of $\{A \in 2^{\mathbb{N}\times\mathbb{N}};\ |\{n;\ A_n \notin U\}| = k\}$ for $k = 0$ (and maybe even for $k \in \mathbb{N}$).

Proposition

Let X be a zero-dimensional Polish space, $\alpha \geq 1$ a countable ordinal and $k \in \mathbb{N}$. If $U \subseteq X$ is $\mathbf{\Pi}^0_\alpha$-complete set, then the following holds. It is well-known that U is G_δ-complete.

- *The set $\{x \in X^{\mathbb{N}} :\ \forall n \in \mathbb{N}\ x_n \notin U\}$ is $\mathbf{\Pi}^0_{\alpha+1}$-complete.*
- *The set $\{x \in X^{\mathbb{N}} :\ |\{n \in \mathbb{N} :\ x_n \in U\}| = k\}$ is $D_2(\mathbf{\Pi}^0_{\alpha+1})$-complete.*

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \ \forall m_0 \in \mathbb{N}\ \exists m \geq m_0 : \ x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),
- $C([0, \omega \cdot k]) \Leftrightarrow |\{n;\ A_n \in U\}| = k - 1$,
- $C([0, \omega^2]) \Leftrightarrow |\{n;\ A_n \notin U\}| = \omega$.

Hence, we need to estimate complexity of $\{A \in 2^{\mathbb{N}\times\mathbb{N}};\ |\{n;\ A_n \notin U\}| = k\}$ for $k = 0$ (and maybe even for $k \in \mathbb{N}$).

Proposition

Let X be a zero-dimensional Polish space, $\alpha \geq 1$ a countable ordinal and $k \in \mathbb{N}$. If $U \subseteq X$ is $\mathbf{\Pi}^0_\alpha$-complete set, then the following holds. It is well-known that U is G_δ-complete.

- *The set $\{x \in X^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ x_n \notin U\}$ is $\mathbf{\Pi}^0_{\alpha+1}$-complete.*
- *The set $\{x \in X^{\mathbb{N}} : \ |\{n \in \mathbb{N} : \ x_n \in U\}| = k\}$ is $D_2(\mathbf{\Pi}^0_{\alpha+1})$-complete.*

Hence, the sets

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ \forall n \in \mathbb{N}\ A_n \notin U\}, \quad P_3^k := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} : \ |\{n \in \mathbb{N} : \ A_n \in U\}| = k\}$$

are $F_{\sigma,\delta}$-complete and $D_2(F_{\sigma,\delta})$-complete, respectively (for every $k \geq 1$).

Lower bound - summary

Summary: Consider the set $U := \{x \in 2^{\mathbb{N}} : \forall m_0 \in \mathbb{N}\, \exists m \geq m_0 : x_m = 1\}$ (infinitely many 1's in a sequence). For $A \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ consider $A_n \in 2^{\mathbb{N}}$ defined by $A_n(m) := a_{n,m}$.

We have continuous mapping $\varphi : 2^{\mathbb{N}\times\mathbb{N}} \to \mathcal{B}$ such that $X_{\varphi(A)}$ is isometric to

- $c \Leftrightarrow A_n \notin U$ for every $n \in \mathbb{N}$ (finitely many 1's in each row),
- $C([0, \omega \cdot k]) \Leftrightarrow |\{n;\ A_n \in U\}| = k - 1$,
- $C([0, \omega^2]) \Leftrightarrow |\{n;\ A_n \notin U\}| = \omega$.

Hence, we need to estimate complexity of $\{A \in 2^{\mathbb{N}\times\mathbb{N}};\ |\{n;\ A_n \notin U\}| = k\}$ for $k = 0$ (and maybe even for $k \in \mathbb{N}$).

Proposition

Let X be a zero-dimensional Polish space, $\alpha \geq 1$ a countable ordinal and $k \in \mathbb{N}$. If $U \subseteq X$ is $\mathbf{\Pi}^0_\alpha$-complete set, then the following holds. It is well-known that U is G_δ-complete.

- *The set $\{x \in X^{\mathbb{N}} :\ \forall n \in \mathbb{N}\ x_n \notin U\}$ is $\mathbf{\Pi}^0_{\alpha+1}$-complete.*
- *The set $\{x \in X^{\mathbb{N}} :\ |\{n \in \mathbb{N} :\ x_n \in U\}| = k\}$ is $D_2(\mathbf{\Pi}^0_{\alpha+1})$-complete.*

Hence, the sets

$$P_3 := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} :\ \forall n \in \mathbb{N}\ A_n \notin U\}, \quad P_3^k := \{A \in (2^{\mathbb{N}})^{\mathbb{N}} :\ |\{n \in \mathbb{N} :\ A_n \in U\}| = k\}$$

are $F_{\sigma,\delta}$-complete and $D_2(F_{\sigma,\delta})$-complete, respectively (for every $k \geq 1$).

Moreover, $P_3 \leq_W \langle c \rangle_\equiv$ and $P_3^k \leq_W \langle C([0, \omega \cdot (k+1)]) \rangle_\equiv$.

PART SEVEN: Complexity of the isometry class of some $C(K)$ spaces - main results and consequences

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Comments on the proof:

- *Succesor step:* given a “matrix” $A \in X$ from the previous step we obtain spaces isometric to $C(K_A)$ with $K_A \in \{\omega^\alpha \cdot k, \omega^{\alpha+1}\}$.

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Comments on the proof:

- *Succesor step:* given a "matrix" $A \in X$ from the previous step we obtain spaces isometric to $C(K_A)$ with $K_A \in \{\omega^\alpha \cdot k, \omega^{\alpha+1}\}$. Now consider "matrix" $A^+ \in X^{\mathbb{N}}$ and assign to it Banach space $C(K_{A^+})$, where K_{A^+} is one-point compactification of $\bigcup K_{A^+_n}$. Then

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Comments on the proof:

- *Succesor step:* given a “matrix” $A \in X$ from the previous step we obtain spaces isometric to $C(K_A)$ with $K_A \in \{\omega^\alpha \cdot k, \omega^{\alpha+1}\}$. Now consider “matrix” $A^+ \in X^{\mathbb{N}}$ and assign to it Banach space $C(K_{A^+})$, where K_{A^+} is one-point compactification of $\bigcup K_{A_n^+}$. Then

  - $K_{A^+} \approx [0, \omega^{\alpha+1}]$ iff $K_{A_n} \not\approx [0, \omega^{\alpha+1}]$ for every n,

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Comments on the proof:

- *Succesor step:* given a “matrix” $A \in X$ from the previous step we obtain spaces isometric to $C(K_A)$ with $K_A \in \{\omega^\alpha \cdot k, \omega^{\alpha+1}\}$. Now consider “matrix” $A^+ \in X^{\mathbb{N}}$ and assign to it Banach space $C(K_{A^+})$, where K_{A^+} is one-point compactification of $\bigcup K_{A_n^+}$. Then

  - $K_{A^+} \approx [0, \omega^{\alpha+1}]$ iff $K_{A_n} \not\approx [0, \omega^{\alpha+1}]$ for every n,

  - $K_{A^+} \approx [0, \omega^{\alpha+1}\dot{(}k+1)]$ iff $K_{A_n} \approx [0, \omega^{\alpha+1}]$ for k-many n’s,

Main results

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Comments on the proof:

- *Succesor step:* given a "matrix" $A \in X$ from the previous step we obtain spaces isometric to $C(K_A)$ with $K_A \in \{\omega^\alpha \cdot k, \omega^{\alpha+1}\}$. Now consider "matrix" $A^+ \in X^{\mathbb{N}}$ and assign to it Banach space $C(K_{A^+})$, where K_{A^+} is one-point compactification of $\bigcup K_{A_n^+}$. Then

  - $K_{A^+} \approx [0, \omega^{\alpha+1}]$ iff $K_{A_n} \not\approx [0, \omega^{\alpha+1}]$ for every n,

  - $K_{A^+} \approx [0, \omega^{\alpha+1} \cdot (k+1)]$ iff $K_{A_n} \approx [0, \omega^{\alpha+1}]$ for k-many n's,

  - otherwise $K_{A^+} \approx [0, \omega^{\alpha+2}]$.

Main results

Theorem (Main result)

Let β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The isometry class of $\mathcal{C}(\omega^{\beta+n})$ is a $\mathbf{\Pi}^0_{\beta+2n+1}$-complete set.*
- *The isometry class of $\mathcal{C}(\omega^{\beta+n} \cdot k)$ is a $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete set for every $k \in \mathbb{N}$ with $k \geq 2$.*
- *The isometry class of $\mathcal{C}(2^{\mathbb{N}})$ is a $\mathbf{\Pi}^0_3$-complete set.*

Comments on the proof:

- *Succesor step:* given a "matrix" $A \in X$ from the previous step we obtain spaces isometric to $C(K_A)$ with $K_A \in \{\omega^\alpha \cdot k, \omega^{\alpha+1}\}$. Now consider "matrix" $A^+ \in X^{\mathbb{N}}$ and assign to it Banach space $C(K_{A^+})$, where K_{A^+} is one-point compactification of $\bigcup K_{A_n^+}$. Then

  - $K_{A^+} \approx [0, \omega^{\alpha+1}]$ iff $K_{A_n} \not\approx [0, \omega^{\alpha+1}]$ for every n,

  - $K_{A^+} \approx [0, \omega^{\alpha+1}\dot{(}k+1)]$ iff $K_{A_n} \approx [0, \omega^{\alpha+1}]$ for k-many n's,

  - otherwise $K_{A^+} \approx [0, \omega^{\alpha+2}]$.
- *Limit step:* Most work done by Cenzer and Mauldin, then proceed as above .. see the next slide.

Limit step

Comments on the proof - limit step: For the limit step, it suffices to prove it for homeomorphism classes:

There exists a continuous reduction "$K \mapsto C(K)$" from $\mathcal{K}(2^{\mathbb{N}}) = \{K \subset 2^{\mathbb{N}};\ K \text{ is compact}\}$ to $\mathcal{P}$ and F_σ-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.

Limit step

Comments on the proof - limit step: For the limit step, it suffices to prove it for homeomorphism classes:

There exists a continuous reduction "$K \mapsto C(K)$" from $\mathcal{K}(2^{\mathbb{N}}) = \{K \subset 2^{\mathbb{N}};\ K \text{ is compact}\}$ to $\mathcal{P}$ and F_σ-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.

If $f : X \to Y$ is F_σ-measurable and $f^{-1}(Z)$ is $\mathbf{\Gamma}$-hard for $\mathbf{\Gamma}$ being an infinite Borel class, then Z is $\mathbf{\Gamma}$-hard.

Limit step

Comments on the proof - limit step: For the limit step, it suffices to prove it for homeomorphism classes:

There exists a continuous reduction "$K \mapsto C(K)$" from $\mathcal{K}(2^{\mathbb{N}}) = \{K \subset 2^{\mathbb{N}};\ K \text{ is compact}\}$ to $\mathcal{P}$ and F_σ-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.

If $f : X \to Y$ is F_σ-measurable and $f^{-1}(Z)$ is $\mathbf{\Gamma}$-hard for $\mathbf{\Gamma}$ being an infinite Borel class, then Z is $\mathbf{\Gamma}$-hard.

Cenzer, Mauldin 83: For any countable limit ordinal λ and any $B \in \mathbf{\Pi}^0_\lambda(\mathbb{N}^{\mathbb{N}})$ there exists a continuous reduction $H : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ of B to $\langle[0, \omega^\lambda]\rangle_\sim$ such that $H(x)$ is infinite compact set, homeomorphic to a subset of $[0, \omega^\lambda]$ for every $x \in \mathbb{N}^{\mathbb{N}}$.

Comments on the proof - limit step: For the limit step, it suffices to prove it for homeomorphism classes:

There exists a continuous reduction "$K \mapsto C(K)$" from $\mathcal{K}(2^{\mathbb{N}}) = \{K \subset 2^{\mathbb{N}};\ K \text{ is compact}\}$ to $\mathcal{P}$ and F_σ-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.

If $f : X \to Y$ is F_σ-measurable and $f^{-1}(Z)$ is $\mathbf{\Gamma}$-hard for $\mathbf{\Gamma}$ being an infinite Borel class, then Z is $\mathbf{\Gamma}$-hard.

Cenzer, Mauldin 83: For any countable limit ordinal λ and any $B \in \mathbf{\Pi}^0_\lambda(\mathbb{N}^{\mathbb{N}})$ there exists a continuous reduction $H : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ of B to $\langle [0, \omega^\lambda] \rangle_\sim$ such that $H(x)$ is infinite compact set, homeomorphic to a subset of $[0, \omega^\lambda]$ for every $x \in \mathbb{N}^{\mathbb{N}}$.

And now we use similar idea with one point compactification to improve from $\mathbf{\Pi}^0_\lambda$ to $\mathbf{\Pi}^0_{\lambda+1}$ and proceed inductively ..

Comments on the proof - limit step: For the limit step, it suffices to prove it for homeomorphism classes:

There exists a continuous reduction "$K \mapsto C(K)$" from $\mathcal{K}(2^{\mathbb{N}}) = \{K \subset 2^{\mathbb{N}};\ K \text{ is compact}\}$ to $\mathcal{P}$ and F_σ-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.

If $f : X \to Y$ is F_σ-measurable and $f^{-1}(Z)$ is $\mathbf{\Gamma}$-hard for $\mathbf{\Gamma}$ being an infinite Borel class, then Z is $\mathbf{\Gamma}$-hard.

Cenzer, Mauldin 83: For any countable limit ordinal λ and any $B \in \mathbf{\Pi}^0_\lambda(\mathbb{N}^{\mathbb{N}})$ there exists a continuous reduction $H : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ of B to $\langle[0,\omega^\lambda]\rangle_\sim$ such that $H(x)$ is infinite compact set, homeomorphic to a subset of $[0,\omega^\lambda]$ for every $x \in \mathbb{N}^{\mathbb{N}}$.

And now we use similar idea with one point compactification to improve from $\mathbf{\Pi}^0_\lambda$ to $\mathbf{\Pi}^0_{\lambda+1}$ and proceed inductively ..

Theorem

Let X be an uncountable metrizable compact space, β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The homeomorphism class of $[0,\omega^{\beta+n}]$ in $\mathcal{K}(X)$ is $\mathbf{\Pi}^0_{\beta+2n+1}$-complete.*
- *The homeomorphism class of $[0,\omega^{\beta+n}\cdot k]$ in $\mathcal{K}(X)$ is $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete for every $k \in \mathbb{N}$ with $k \geq 2$.*

Comments on the proof - limit step: For the limit step, it suffices to prove it for homeomorphism classes:

There exists a continuous reduction "$K \mapsto C(K)$" from $\mathcal{K}(2^{\mathbb{N}}) = \{K \subset 2^{\mathbb{N}};\ K \text{ is compact}\}$ to $\mathcal{P}$ and F_σ-measurable reduction from $\mathcal{P}_\infty$ to $\mathcal{B}$.

If $f : X \to Y$ is F_σ-measurable and $f^{-1}(Z)$ is $\mathbf{\Gamma}$-hard for $\mathbf{\Gamma}$ being an infinite Borel class, then Z is $\mathbf{\Gamma}$-hard.

Cenzer, Mauldin 83: For any countable limit ordinal λ and any $B \in \mathbf{\Pi}^0_\lambda(\mathbb{N}^{\mathbb{N}})$ there exists a continuous reduction $H : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ of B to $\langle[0,\omega^\lambda]\rangle_\sim$ such that $H(x)$ is infinite compact set, homeomorphic to a subset of $[0,\omega^\lambda]$ for every $x \in \mathbb{N}^{\mathbb{N}}$.

And now we use similar idea with one point compactification to improve from $\mathbf{\Pi}^0_\lambda$ to $\mathbf{\Pi}^0_{\lambda+1}$ and proceed inductively ..

Theorem

Let X be an uncountable metrizable compact space, β be either 0 or a countable limit ordinal and let $n \in \mathbb{N} \cup \{0\}$ be such that $\beta + n > 0$.

- *The homeomorphism class of $[0,\omega^{\beta+n}]$ in $\mathcal{K}(X)$ is $\mathbf{\Pi}^0_{\beta+2n+1}$-complete.*
- *The homeomorphism class of $[0,\omega^{\beta+n}\cdot k]$ in $\mathcal{K}(X)$ is $D_2(\mathbf{\Pi}^0_{\beta+2n+1})$-complete for every $k \in \mathbb{N}$ with $k \geq 2$.*

Warning: Isometry class of $C(2^{\mathbb{N}})$ is $F_{\sigma,\delta}$-complete, while homeomorphism class of $2^{\mathbb{N}}$ is G_δ-complete.

Corollary

If λ is a countable limit ordinal and (P) is a property of Banach spaces preserved by linear isometries, which holds for the Banach space $\mathcal{C}(\omega^\lambda)$, but not for $\mathcal{C}(\alpha)$ with $\alpha < \omega^\lambda$, then the set

$$\{\mu \in \mathcal{B} : \ X_\mu \text{ has } (P)\}$$

is $\mathbf{\Pi}^0_\lambda$-hard.

Corollary

If λ is a countable limit ordinal and (P) is a property of Banach spaces preserved by linear isometries, which holds for the Banach space $\mathcal{C}(\omega^\lambda)$, but not for $\mathcal{C}(\alpha)$ with $\alpha < \omega^\lambda$, then the set

$$\{\mu \in \mathcal{B} : \ X_\mu \textit{ has } (P)\}$$

is $\mathbf{\Pi}^0_\lambda$-hard.

Applications:

- The set $\{\mu \in \mathcal{B} : \ X_\mu \text{ has summable Szlenk index}\}$ is $\mathbf{\Sigma}^0_\omega$-hard. (It was known to be a $\mathbf{\Sigma}^0_{\omega+2}$-set.)

Corollary

If λ is a countable limit ordinal and (P) is a property of Banach spaces preserved by linear isometries, which holds for the Banach space $\mathcal{C}(\omega^\lambda)$, but not for $\mathcal{C}(\alpha)$ with $\alpha < \omega^\lambda$, then the set

$$\{\mu \in \mathcal{B} : \ X_\mu \text{ has } (P)\}$$

is $\mathbf{\Pi}^0_\lambda$-hard.

Applications:

- The set $\{\mu \in \mathcal{B} : \ X_\mu \text{ has summable Szlenk index}\}$ is $\mathbf{\Sigma}^0_\omega$-hard.
  (It was known to be a $\mathbf{\Sigma}^0_{\omega+2}$-set.)
  Proof: For any compact $K \subseteq [0, \omega^\omega]$, it is well-known that $\mathcal{C}(K)$ has a summable Szlenk index if and only if $K \not\approx [0, \omega^\omega]$.

Corollary

If λ is a countable limit ordinal and (P) is a property of Banach spaces preserved by linear isometries, which holds for the Banach space $\mathcal{C}(\omega^\lambda)$, but not for $\mathcal{C}(\alpha)$ with $\alpha < \omega^\lambda$, then the set

$$\{\mu \in \mathcal{B} : \ X_\mu \text{ has } (P)\}$$

is $\mathbf{\Pi}^0_\lambda$-hard.

Applications:

- The set $\{\mu \in \mathcal{B} : \ X_\mu$ has summable Szlenk index$\}$ is $\mathbf{\Sigma}^0_\omega$-hard.
  (It was known to be a $\mathbf{\Sigma}^0_{\omega+2}$-set.)

  Proof: For any compact $K \subseteq [0, \omega^\omega]$, it is well-known that $\mathcal{C}(K)$ has a summable Szlenk index if and only if $K \not\simeq [0, \omega^\omega]$.
- For any $\alpha \in [1, \omega_1)$, the set $\{\mu \in \mathcal{B} : \ Sz(X_\mu) \leq \omega^\alpha\}$ is $\mathbf{\Sigma}^0_{\omega^\alpha}$-hard.
  (It was known to be a $\mathbf{\Pi}^0_{\omega^\alpha+1}$-set.)

Further applications

Corollary

If λ is a countable limit ordinal and (P) is a property of Banach spaces preserved by linear isometries, which holds for the Banach space $\mathcal{C}(\omega^\lambda)$, but not for $\mathcal{C}(\alpha)$ with $\alpha < \omega^\lambda$, then the set

$$\{\mu \in \mathcal{B} : \ X_\mu \textit{ has } (P)\}$$

is $\mathbf{\Pi}^0_\lambda$-hard.

Applications:

- The set $\{\mu \in \mathcal{B} : \ X_\mu \text{ has summable Szlenk index}\}$ is $\mathbf{\Sigma}^0_\omega$-hard. (It was known to be a $\mathbf{\Sigma}^0_{\omega+2}$-set.)

  Proof: For any compact $K \subseteq [0, \omega^\omega]$, it is well-known that $\mathcal{C}(K)$ has a summable Szlenk index if and only if $K \not\simeq [0, \omega^\omega]$.
- For any $\alpha \in [1, \omega_1)$, the set $\{\mu \in \mathcal{B} : \ Sz(X_\mu) \leq \omega^\alpha\}$ is $\mathbf{\Sigma}^0_{\omega^\alpha}$-hard. (It was known to be a $\mathbf{\Pi}^0_{\omega^\alpha+1}$-set.)

  Proof: For any compact $K \subseteq [0, \omega^{\omega^\alpha}]$, it is well-known that $Sz(\mathcal{C}(K)) \leq \omega^\alpha$ if and only if $K \not\simeq [0, \omega^{\omega^\alpha}]$.

Further applications

Corollary

If λ is a countable limit ordinal and (P) is a property of Banach spaces preserved by linear isometries, which holds for the Banach space $\mathcal{C}(\omega^\lambda)$, but not for $\mathcal{C}(\alpha)$ with $\alpha < \omega^\lambda$, then the set

$$\{\mu \in \mathcal{B} : \; X_\mu \textit{ has } (P)\}$$

is $\mathbf{\Pi}^0_\lambda$-hard.

Applications:

- The set $\{\mu \in \mathcal{B} : \; X_\mu \text{ has summable Szlenk index}\}$ is $\mathbf{\Sigma}^0_\omega$-hard.
  (It was known to be a $\mathbf{\Sigma}^0_{\omega+2}$-set.)

  Proof: For any compact $K \subseteq [0, \omega^\omega]$, it is well-known that $\mathcal{C}(K)$ has a summable Szlenk index if and only if $K \not\simeq [0, \omega^\omega]$.
- For any $\alpha \in [1, \omega_1)$, the set $\{\mu \in \mathcal{B} : \; Sz(X_\mu) \leq \omega^\alpha\}$ is $\mathbf{\Sigma}^0_{\omega^\alpha}$-hard.
  (It was known to be a $\mathbf{\Pi}^0_{\omega^\alpha+1}$-set.)

  Proof: For any compact $K \subseteq [0, \omega^{\omega^\alpha}]$, it is well-known that $Sz(\mathcal{C}(K)) \leq \omega^\alpha$ if and only if $K \not\simeq [0, \omega^{\omega^\alpha}]$.

THANK YOU FOR YOUR ATTENTION ;)