

Adaptive iterative linearized Galerkin with non-conforming methods

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Adaptive iterative linearization Galerkin methods for nonlinear problems, Heid & Wihler, 2020 & On the convergence of adaptive iterative linearized Galerkin methods, Heid & Wihler, 2020 — analysis of the iterative linearized Galerkin (ILG) method for a strongly monotone & Lipschitz continuous nonlinear operator equation

- Finite dimensional subspace is a subspace of the continuous space (conforming)
- Operator on subspace restriction of original operator to finite dimensional subspace

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- Finite dimensional subspace is a subspace of the continuous space (conforming)
- Operator on subspace restriction of original operator to finite dimensional subspace

So what happens if we break some of these assumptions?

- Finite dimensional space is not a subspace (non-conforming)
- Finite dimensional operator is not just a restriction, but a different operator
- Additional terms/operators (e.g. stabilization)



Hilbert space X , with inner product $(\cdot, \cdot)_X$ inducing the norm $\|\cdot\|_X$.

Nonlinear operator $F : X \rightarrow X^*$, where X^* dual space of X — find $u \in X$ such that

$$F(u) = 0 \quad \text{in } X^*.$$

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With weak formulation — find $u \in X$ such that

$$\langle F(u), v \rangle_{X^* \times X} = 0 \quad \text{for all } v \in X. \quad (\text{NLE})$$

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$$\langle F(u), v \rangle_{X^* \times X} = 0 \quad \text{for all } v \in X. \quad (\text{NLE})$$

Assumptions:

(F1) F is **Lipschitz continuous**: $\exists L_F > 0$ such that

$$|\langle F(u) - F(v), w \rangle_{X^* \times X}| \leq L_F \|u - v\|_X \|w\|_X \quad \forall u, v, w \in X$$

(F2) F is **strongly monotone**: $\exists M_F > 0$ such that

$$M_F \|u - v\|_X \leq \langle F(u) - F(v), u - v \rangle_{X^* \times X} \quad \forall u, v \in X$$



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$$u^{(n+1)} = u^{(n)} - A[u^{(n)}]^{-1}F(u^{(n)}), \quad n \geq 0$$

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Alternatively:

$$A[u^{(n)}]u^{(n+1)} = A[u^{(n)}]u^{(n)} - F(u^{(n)}), \quad n \geq 0,$$

or, defining $f : X \rightarrow X^*$, as $f(u) := A[u^{(n)}]u^{(n)} - F(u^{(n)})$,

$$A[u^{(n)}]u^{(n+1)} = f(u), \quad n \geq 0. \tag{IL}$$

Define bilinear form, for fixed $u \in X$,

$$a(u; v, w) := \langle A[u]v, w \rangle_{X^* \times X}, \quad u, v \in X;$$

then, given $u^{(n)} \in X$ weak formulation of (IL) is to find $u^{(n+1)} \in X$ such that

$$a(u^{(n)}; u^{(n+1)}, v) = \langle f(u^{(n)}), v \rangle_{X^* \times X} \quad \forall v \in X. \quad (\text{WF})$$

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$$a(u^{(n)}; u^{(n+1)}, v) = \langle f(u^{(n)}), v \rangle_{X^* \times X} \quad \forall v \in X. \quad (\text{WF})$$

Assumptions:

(A1) $a(u; \cdot, \cdot)$ is coercive: $\exists \alpha > 0$ such that

$$a(u; v, v) \geq \alpha \|v\|_X^2 \quad \forall v \in X.$$

(A1) $a(u; \cdot, \cdot)$ is bounded: $\exists \beta > 0$ such that

$$a(u; v, w) \leq \beta \|v\|_X \|w\|_X \quad \forall v, w \in X.$$

Due to assumptions a unique solution $u \in X$ of (WF) exists by Lax-Milgram.

Zarantonello For $\varepsilon > 0$ sufficiently small (depending on L_F and M_F) set $A[u] := \varepsilon^{-1} J_X$, where J_X is the Riesz-isomorphism:

$$(u^{(n+1)}, v)_X = (u^{(n)}, v)_X - \varepsilon \langle F(u^{(n)}), v \rangle_{X^* \times X} \quad \forall v \in X, n \geq 0$$

Kačanov Defining $A[u]$ such that $f(u) = A[u]u - F(u) =: g$ independent of u :

$$\langle A[u^{(n)}]u^{(n+1)}, v \rangle_{X^* \times X} = \langle g, v \rangle_{X^* \times X} \quad \forall v \in X, n \geq 0$$

(Damped) Newton For damping parameter ε_n^{-1} , set $A[u] = \varepsilon_n^{-1} F'(u)$, where F' is the Gâteaux derivative of F :

$$\langle F'(u^{(n)})u^{(n+1)}, v \rangle_{X^* \times X} = \langle F'(u^{(n)})u^{(n)}, v \rangle_{X^* \times X} - \varepsilon \langle F(u^{(n)}), v \rangle_{X^* \times X} \quad \forall v \in X, n \geq 0$$

Consider finite dimensional space $X_N \subset X$; then, the Galerkin approximation of (NLE) is: find $u_N \in X_N$ such that

$$\langle F(u_N), v \rangle_{X^* \times X^*} = 0 \quad \text{for all } v \in X_N. \quad (\text{FD})$$

The iterative linearized Galerkin (ILG) method:

for given initial guess $u_N^{(0)} \in X_N$ find $u_N^{(n+1)} \in X_N$ such that

$$a(u_N^{(n)}; u_N^{(n+1)}, v) = \langle f(u_N^{(n)}), v \rangle_{X^* \times X^*} \quad \forall v \in X_N \quad (\text{ILG})$$

Consider finite dimensional space $X_N \sim X$; then, the **non-conforming** Galerkin approximation of (NLE) is: find $u_N \in X_N$ such that

$$\langle F(u_N), v \rangle_{X_N^* \times X_N^*} = 0 \quad \text{for all } v \in X_N. \quad (\text{FD})$$

The **non-conforming** iterative linearized Galerkin (ILG) method:

for given initial guess $u_N^{(0)} \in X_N$ find $u_N^{(n+1)} \in X_N$ such that

$$a_N(u_N^{(n)}; u_N^{(n+1)}, v) = \langle f(u_N^{(n)}), v \rangle_{X_N^* \times X_N^*} \quad \forall v \in X_N \quad (\text{ILG})$$

where

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where

$$\begin{aligned} a_N(u_N; v, w) &:= \langle A_N[u_N]v, w \rangle_{X_N^* \times X_N}, & u_N, v, w &\in X_N, \\ f_N(u_N) &:= A_N[u_N]u_N - F_N(u_N), & u_N &\in X_N, \end{aligned}$$

Consider finite dimensional space $X_N \sim X$; then, the **non-conforming** Galerkin approximation of (NLE) is: find $u_N \in X_N$ such that

$$\langle F_N(u_N) + S_N(u_N), v \rangle_{X_N^* \times X_N^*} = 0 \quad \text{for all } v \in X_N. \quad (\text{FD})$$

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$$\begin{aligned} a_N(u_N; v, w) &:= \langle A_N[u_N]v, w \rangle_{X_N^* \times X_N}, & u_N, v, w &\in X_N, \\ f_N(u_N) &:= A_N[u_N]u_N - F_N(u_N) - S_N(u_N), & u_N &\in X_N, \\ \langle S_N(v), v \rangle_{X_N^* \times X_N} &\geq 0 & \forall v &\in X_N. \end{aligned}$$

We additionally require for X_N a **semi-norm** $|\cdot|_{X_N}$, inner product $(\cdot, \cdot)_{X_N}$, and norm $\|\cdot\|_{X_N}$ defined for $X \cup X_N$ such that

$$\|u\|_{X_N} \equiv \|u\|_X \quad \forall u \in X.$$

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Additionally we assume that for $v \in X_N$, $\exists C_0, C_1, \tilde{M}_F > 0$ such that

$$C_0 |v|_{X_N} \leq \|v\|_{X_N} \leq C_1 \left(|v|_{X_N} + \tilde{M}_F^{-1} \langle S_N(v), v \rangle_{X_N^* \times X_N} \right) \quad (\text{S})$$

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and **one** of the following holds $\forall v, w \in X_N$:

$$\|v - w\|_{X_N} \leq C_1 \left(|v - w|_{X_N} + \tilde{M}_F^{-1} \langle S_N(v) - S_N(w), v - w \rangle_{X_N^* \times X_N} \right),$$

$$\langle S_N(u - v), u - v \rangle_{X_N^* \times X_N} \leq \langle S_N(v) - S_N(w), v - w \rangle_{X_N^* \times X_N},$$

S_N is linear

Assumptions:

(FN1) F_N is **Lipschitz continuous**: $\exists \tilde{L}_F > 0$ such that

$$|\langle F_N(u) - F_N(v), w \rangle_{X_N^* \times X_N}| \leq \tilde{L}_F |u - v|_{X_N} |w|_{X_N} \quad \forall u, v, w \in X$$

(FN2) F_N is **strongly monotone**: $\exists \tilde{M}_F > 0$ such that

$$\tilde{M}_F |u - v|_{X_N} \leq \langle F_N(u) - F_N(v), u - v \rangle_{X^* \times X} \quad \forall u, v \in X$$

(AN1) $a_N(u_N; \cdot, \cdot)$ is coercive: $\exists \tilde{\alpha} > 0$ such that

$$a_N(u_N; v, v) \geq \tilde{\alpha} |v|_{X_N}^2 \quad \forall v \in X_N.$$

(AN1) $a_N(u_N; \cdot, \cdot)$ is bounded: $\exists \tilde{\beta} > 0$ such that

$$a_N(u_N; v, w) \leq \tilde{\beta} |v|_{X_N} |w|_{X_N} \quad \forall v, w \in X_N.$$

Lemma

For $u_N \in X_N$ solution of (FD) and sequence $\{u_N^{(n)}\}_{n \geq 0}$ generated by (ILG)

$$\|u_N - u_N^{(n)}\|_{X_N} \leq C_{\text{It}} \|u_N^{(n)} - u_N^{(n-1)}\|, \quad C_{\text{It}} = 1 + \tilde{\beta} C_1 \tilde{M}_F^{-1}$$

Proof.

Follows similarly to [Lemma 1, Heid & Wihler, 2020](#). Use triangle inequality and

$$\begin{aligned}
 \frac{\tilde{M}_F}{C_1} \|u_N - u_N^{(n-1)}\|_{X_N}^2 &\leq \tilde{M}_F \|u_N - u_N^{(n-1)}\|_{X_N} + \langle S_N(u_N) - S_N(u_N^{(n-1)}), u_N - u_N^{(n-1)} \rangle_{X_N^* \times X_N} \\
 &\leq \langle F_N(u_N) - F_N(u_N^{(n-1)}), u_N - u_N^{(n-1)} \rangle_{X_N^* \times X_N} \\
 &\quad + \langle S_N(u_N) - S_N(u_N^{(n-1)}), u_N - u_N^{(n-1)} \rangle_{X_N^* \times X_N} \\
 &= \langle F_N(u_N^{(n-1)}) + S_N(u_N^{(n-1)}), u_N^{(n-1)} - u_N \rangle_{X_N^* \times X_N} \\
 &= \langle A_N[u_N^{(n-1)}]u_N^{(n-1)} - A_N[u_N^{(n-1)}]u_N^{(n)}, u_N^{(n-1)} - u_N \rangle_{X_N^* \times X_N} \\
 &= a_N(u_N^{(n-1)}; u_N^{(n-1)} - u_N^{(n)}, u_N^{(n-1)} - u_N) \\
 &\leq \tilde{\beta} \|u_N^{(n-1)} - u_N^{(n)}\| \|u_N^{(n-1)} - u_N\|.
 \end{aligned}$$



Consider sequence of spaces X_N , $N \geq 0$, which are not guaranteed to be hierarchical, with $\dim(X_0) < \dim(X_1) < \dots < \dim(X_N) < \dots$.

For $N \geq 0$ we assume there exists a computable error indicator $\eta_N : X_N \rightarrow \mathbb{R}_+$ with the assumptions that

(I1) for all $u, v \in X_N$

$$|\eta_N(u) - \eta_N(v)| \leq C_{\eta,1} \|u - v\|_{X_N},$$

(I2) the error indicator is a bound for $u_N \in X_N$ from (FD); i.e., $\exists C_\eta \geq 0$ such that

$$\|u - u_N\|_{X_N} \leq C_{\eta,2} \eta_N(u_N).$$

Lemma

We assume that $\exists \lambda \in (0, C_\lambda^{-1})$, where $C_\lambda = C_{\text{It}} C_{\eta,1}$, such that

$$\|u_N^{(n)} - u_N^{(n-1)}\| \leq \lambda \eta_N(u_N^{(n)}).$$

Then,

$$\|u_N - u_N^{(n)}\| \leq \lambda C_{\text{It}} \min\{\eta_N(u_N^{(n)}), (1 - \lambda C_\lambda)^{-1} \eta_N(u_N)\}$$

and

$$(1 - \lambda C_\lambda) \eta_N(u_N^{(n)}) \leq \eta_N(u_N) \leq (1 + \lambda C_\lambda) \eta_N(u_N^{(n)}).$$

Remark

The assumption can be met by the adaptive ILG algorithm.

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$$(1 - \lambda C_\lambda) \eta_N(u_N^{(n)}) \leq \eta_N(u_N) \leq (1 + \lambda C_\lambda) \eta_N(u_N^{(n)}).$$

Proof.

Identical to Lemma 4, Heid & Wihler, 2020; cf. Gartner, Haberl, Praetorius, Stifner, 2018. □

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and

$$(1 - \lambda C_\lambda) \eta_N(u_N^{(n)}) \leq \eta_N(u_N) \leq (1 + \lambda C_\lambda) \eta_N(u_N^{(n)}).$$

Corollary

$$\|u - u_N^{(n)}\| \leq (1 + \lambda C_\lambda (C_{\eta,1} + 1)) \eta_N(u_N^{(n)})$$

Algorithm (Heid & Wihler, 2020)

For tolerance TOL and $\lambda > 0$, $N = 0$, initial space X_N and guess $u_0^{(0)} \in X_0$.

while $\eta_N(u_N^{(0)}) > \text{TOL}$ **do**

$n \leftarrow 1$

Compute $u_N^{(n)}$ from single step of (ILG)

while $\|u_N^{(n)} - u_N^{(n-1)}\| > \lambda \eta_N(u_N^{(n)})$ **do**

Compute $u_N^{(n+1)}$ from single step of (ILG)

$n \leftarrow n + 1$

end while

$u_N^F \leftarrow u_N^{(n)} \in X_N$

Enrich X_N based on $\eta_N(u_N^F)$ to obtain X_{N+1}

$u_{N+1}^{(0)} \leftarrow P_{N+1} u_N^F$

$\triangleright (P_{N+1} : X_N \rightarrow X_{N+1} \text{ is a prolongation operator})$

$N \leftarrow N + 1$

end while

Open question — does the adaptive ILG algorithm converge for non-conforming methods?

Requires conditions on

- the error indicators η_N and η_{N+1} for convergence (e.g., meeting some of the **Axioms of adaptivity** [Carstensen, Feischl, Page, Praetorius, 2013])
- the prolongation operator $P_{N+1} : X_N \rightarrow X_{N+1}$ (e.g., a bound on $\|u_N - P_{N+1}u_N\|$)

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Proof for conforming case (cf. **Heid & Wihler, 2020** & **Gartner, Haberl, Praetorius, Stifner, 2018**) use the fact that $X_0 \subset X_1 \subset \dots \subset X_N$ and proves result on the space

$$X_\infty = \bigcup_{N \geq 0} X_N.$$

Example — Non-conforming VEM



Consider the PDE

$$\begin{aligned} -\nabla \cdot (\mu(|\nabla u|)\nabla u) &= f, & \text{in } \Omega \subset \mathbb{R}^2, \\ u &= 0, & \text{on } \partial\Omega, \end{aligned}$$

where $\exists m_\mu, M_\mu > 0$ such that

$$m_\mu(t-s) \leq \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \leq M_\mu(t-s), \quad t \geq s \geq 0, \quad \mathbf{x} \in \bar{\Omega}.$$

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Let $X = H_0^1(\Omega)$, $\|\cdot\|_X = \|\cdot\|_{H^1}$, and

$$\langle F(u), v \rangle_{X^* \times X} = \int_{\Omega} \mu(|\nabla u|)\nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\Omega} f v \, d\mathbf{x}.$$

Due to the assumptions on μ (F1) and (F2) can be shown.

Example — Non-conforming VEM



We will consider a non-conforming VEM fomulation [Ayuso de Dios, Lipnikov, Manzini, 2016].

First, we define the broken Sobolev spaces for a mesh $\mathcal{T}_N$ for $\ell \geq 1$

$$H^1(\mathcal{T}_N) := \{v \in L^2(\Omega) : v|_K \in H^1(K), \quad \forall K \in \mathcal{T}_N\},$$

$$H^{1,\text{nc}}(\mathcal{T}_N) := \left\{ v \in H^1(\mathcal{T}_N) : \int_e \llbracket v \rrbracket \cdot \mathbf{n}_e p \, ds = 0 \, \forall p \in \mathbb{P}_{\ell-1}(e), \forall e \in \mathcal{E}_N \right\}.$$

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Given a **local enlarged VEM space**

$$\tilde{V}_{h,\ell}^K := \{v_h \in H^1(K) : \Delta v_h \in \mathbb{P}_\ell(K), v_h|_e \in \mathbb{P}_\ell(e) \forall e \subset \partial K\}$$

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and a **value projection** $\Pi_0^K : \tilde{V}_{h,\ell}^K \rightarrow \mathbb{P}_\ell$ we define the **local virtual element space** $V_{h,\ell}^K$ as

$$V_{h,\ell}^K := \left\{ v_h \in \tilde{V}_{h,\ell}^K : (v_h - \Pi_0^K v_h, p)_K = 0 \quad \forall p \in \mathbb{P}_\ell(K) \setminus \mathbb{P}_{\ell-2}(K) \right\}$$

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$$H^{1,\text{nc}}(\mathcal{T}_N) := \left\{ v \in H^1(\mathcal{T}_N) : \int_e \llbracket v \rrbracket \cdot \mathbf{n}_e p \, ds = 0 \quad \forall p \in \mathbb{P}_{\ell-1}(e), \forall e \in \mathcal{E}_N \right\}.$$

Given a **local enlarged VEM space**

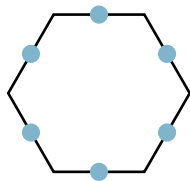
$$\tilde{V}_{h,\ell}^K := \{v_h \in H^1(K) : \Delta v_h \in \mathbb{P}_\ell(K), v_h|_e \in \mathbb{P}_\ell(e) \forall e \subset \partial K\}$$

and a **value projection** $\Pi_0^K : \tilde{V}_{h,\ell}^K \rightarrow \mathbb{P}_\ell$ we define the **local virtual element space** $V_{h,\ell}^K$ as

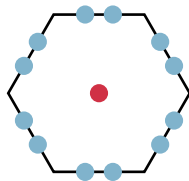
$$V_{h,\ell}^K := \left\{ v_h \in \tilde{V}_{h,\ell}^K : (v_h - \Pi_0^K v_h, p)_K = 0 \quad \forall p \in \mathbb{P}_\ell(K) \setminus \mathbb{P}_{\ell-2}(K) \right\}$$

The **global VEM space**: $X_N := \{v_h \in H^{1,\text{nc}}(\mathcal{T}_N) : v_h|_K \in V_{h,\ell}^K \quad \forall K \in \mathcal{T}_N\}$

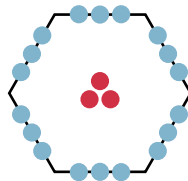
Example — Non-conforming VEM



$\ell = 1$



$\ell = 2$



$\ell = 3$

(D1) The moments of v_h up to order $\ell - 1$ on each edge $e \in \partial K$

$$\frac{1}{|e|} \int_e v_h p \, ds \quad \forall p \in \mathbb{P}_{\ell-1}(e)$$

(D2) For $\ell > 1$, the moments of v_h up to order $\ell - 2$ inside K

$$\frac{1}{|E|} \int_E v_h p \, dx \quad \forall p \in \mathbb{P}_{\ell-2}(K)$$

Example — Non-conforming VEM



Value projection ($\Pi_0^K : \tilde{V}_{h,\ell}^K \rightarrow \mathbb{P}_\ell(K)$) $\Pi_0^K v_h$ linear combination of dofs, and satisfies

$$\int_K \Pi_0^K v_h p \, d\mathbf{x} = \int_K v_h p \, d\mathbf{x} \quad \forall p \in \mathbb{P}_{\ell-4}(K), \quad \text{and} \quad \Pi_0^K q = q \quad \forall q \in \mathbb{P}_\ell(K).$$

Edge projection ($\Pi_0^e : \tilde{V}_{h,\ell}^K \rightarrow \mathbb{P}_{\ell-1}(e)$) $\Pi_0^e v_h$ linear combination of dofs, and satisfies,

$$\int_e \Pi_0^e v_h p \, ds = \int_e v_h p \, ds \quad \forall p \in \mathbb{P}_{\ell-1}(e), \quad \text{and} \quad \Pi_0^e q = q|_e \quad \forall q \in \mathbb{P}_\ell(K).$$

Gradient projection ($\Pi_1^K : \tilde{V}_{h,\ell}^K \rightarrow [\mathbb{P}_{\ell-1}(K)]^2$)

$$\int_K \Pi_1^K v_h \cdot \mathbf{p} \, d\mathbf{x} = - \int_K \Pi_0^K v_h \nabla \cdot \mathbf{p} \, d\mathbf{x} + \sum_{e \in \partial K} \int_e \Pi_0^e v_h \mathbf{p} \cdot \mathbf{n} \, ds \quad \forall \mathbf{p} \in [\mathbb{P}_{\ell-1}(K)]^2.$$

Use CLS for choice of projections: Dedner & Hodson 2024,

Example — Non-conforming VEM



Let $\|\cdot\|_{X_N}$ be the broken H^1 norm, and

$$|v|_{X_N} = \sum_{K \in \mathcal{T}_N} \int_K |\Pi_1^K v|^2 dx,$$

$$(u, v)_{X_N} = \sum_{K \in \mathcal{T}_N} \int_K \Pi_1^K u \cdot \Pi_1^K v dx,$$

$$\langle F_N(u), v \rangle_{X_N^* \times X_N} = \sum_{K \in \mathcal{T}_N} \int_K \mu(|\Pi_1^K u|) \Pi_1^K u \cdot \Pi_1^K v dx - \sum_{K \in \mathcal{T}_N} \int_K f \Pi_0^K v dx,$$

$$\langle S_N(u), v \rangle_{X_N^* \times X_N} = \sum_{K \in \mathcal{T}_N} S_K(u; u - \Pi_0^K u, v - \Pi_0^K v),$$

where $S_K(\cdot; \cdot, \cdot)$ is an **admissable stabilization**.

Example — Non-conforming VEM



Let $\|\cdot\|_{X_N}$ be the broken H^1 norm, and

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where $S_K(\cdot; \cdot, \cdot)$ is an **admissible stabilization**. The admissability condition, plus the fact that Π_0^K is the L^2 -projection into the polynomial space, shows (S). Additionally, due to the assumptions on μ (FN1) and (FN2) can be shown.

Theorem (Conforming Error Bound [C. & Hodson (2026)])

$$\|u - u_N\|_{X_N}^2 \leq C \sum_{K \in \mathcal{T}_N} (\eta_K^2 + \Theta_K^2 + \mathcal{S}_K^2 + \Psi_K^2)$$

where

$$\begin{aligned} \eta_K^2 &:= h_K^2 \|\Pi_0^K f + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^K \nabla u_N|) \mathcal{P}_{\ell-1}^K \nabla u_N\|_{0,K}^2 \\ &\quad + \sum_{e \in \partial K} h_e \|\llbracket \mu_h(|\mathcal{P}_{\ell-1}^K \nabla u_N|) \mathcal{P}_{\ell-1}^K \nabla u_N \rrbracket\|_{0,e}^2, \\ \mathcal{S}_K^2 &:= S_K(u_N; (I - \mathcal{P}_\ell^K)u_N, (I - \mathcal{P}_\ell^K)u_N), \end{aligned}$$

and Θ_K and Ψ_K are data oscillation terms.

Theorem (Non-Conforming Error Bound)

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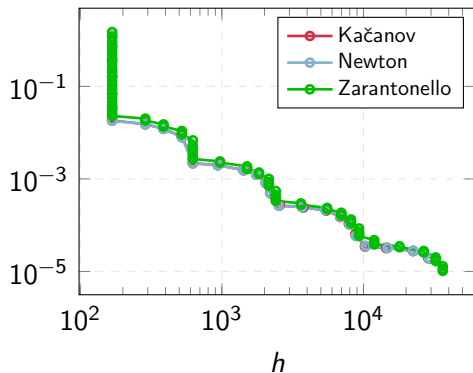
where

$$\begin{aligned} \eta_K^2 &:= h_K^2 \|\Pi_0^K f + \nabla \cdot \mu_h(|\mathcal{P}_{\ell-1}^K \nabla u_N|) \mathcal{P}_{\ell-1}^K \nabla u_N\|_{0,K}^2 \\ &\quad + \sum_{e \in \partial K} h_e \|\llbracket \mu_h(|\mathcal{P}_{\ell-1}^K \nabla u_N|) \mathcal{P}_{\ell-1}^K \nabla u_N \rrbracket\|_{0,e}^2 + \sum_{e \in \partial K} h_e^{-1} \|\llbracket \mathcal{P}_{\ell}^K u_N \rrbracket\|_{0,e}^2, \\ \mathcal{S}_K^2 &:= S_K(u_N; (I - \mathcal{P}_{\ell}^K)u_N, (I - \mathcal{P}_{\ell}^K)u_N), \end{aligned}$$

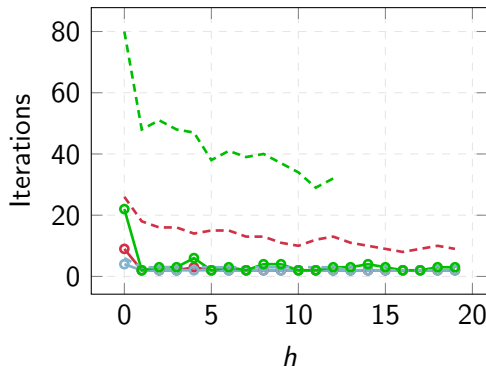
and Θ_K and Ψ_K are data oscillation terms.

$$\Omega = (0, 1)^2 \quad \mu(t) = \frac{1}{2} + \frac{1}{1+t^2}$$

$$u = \sin(\pi x) \sin(\pi y)$$



Error



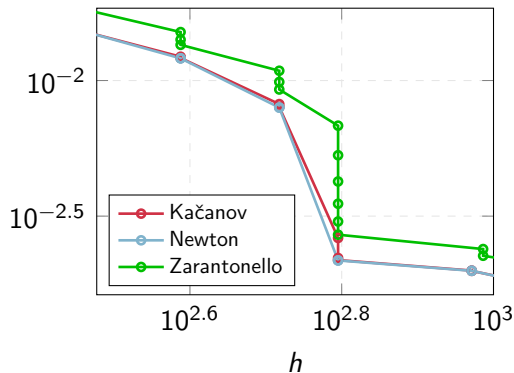
#Iterations

Numerical Experiments (I)

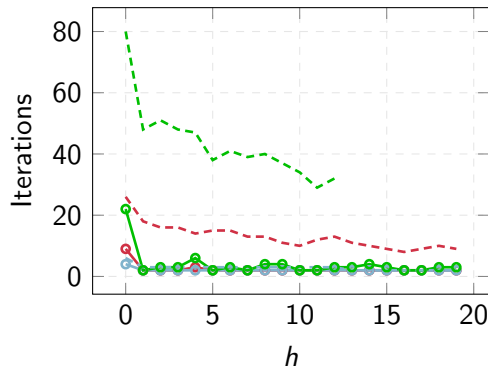


$$\Omega = (0, 1)^2 \quad \mu(t) = \frac{1}{2} + \frac{1}{1+t^2}$$

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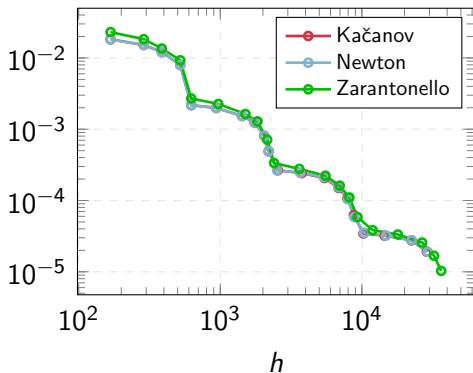
Error



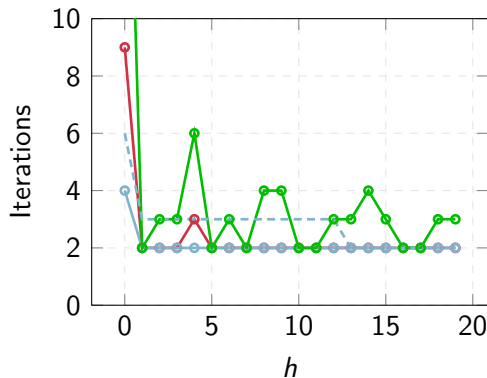
#Iterations

$$\Omega = (0, 1)^2 \quad \mu(t) = \frac{1}{2} + \frac{1}{1+t^2}$$

$$u = \sin(\pi x) \sin(\pi y)$$

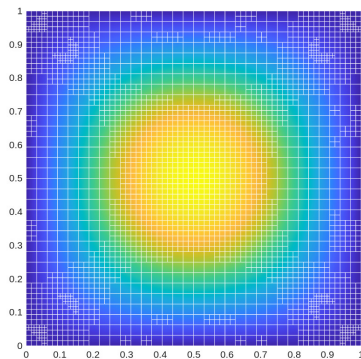


Error



#Iterations

$$\Omega = (0, 1)^2 \quad \mu(t) = \frac{1}{2} + \frac{1}{1 + t^2} \quad u = \sin(\pi x) \sin(\pi y)$$



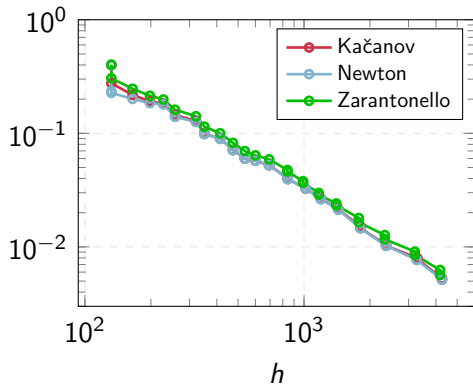
$$u_N \in X_{17} \text{ on } \mathcal{T}_{17}$$

Numerical Experiments (II)

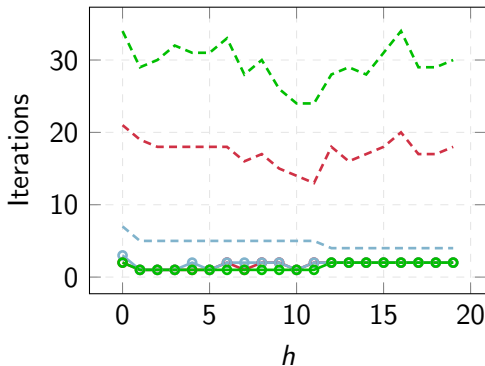


Use the L-shape domain $\Omega = (-1, 1)^2 \setminus ([0, 1] \times (-1, 0])$ and

$$\mu(t) = 1 + e^{-t} \quad u = r^{2/3} \sin(2\varphi/3) x(x^2 - 1) y(y^2 - 1)$$



Error



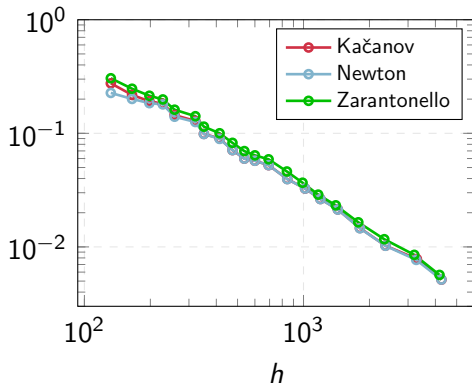
#Iterations

Numerical Experiments (II)

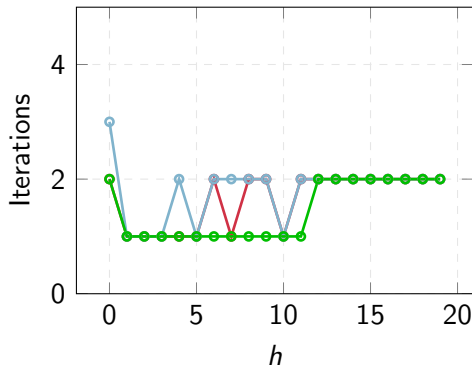


Use the L-shape domain $\Omega = (-1, 1)^2 \setminus ([0, 1] \times (-1, 0])$ and

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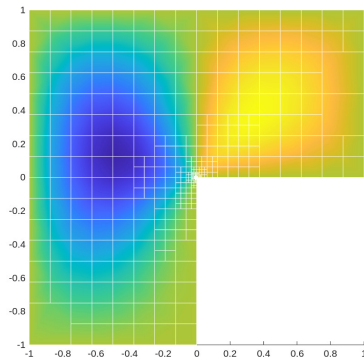
Error



#Iterations

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$$\mu(t) = 1 + e^{-t} \quad u = r^{2/3} \sin(2\varphi/3) x(x^2 - 1) y(y^2 - 1)$$



$$u_N \in X_{19} \text{ on } \mathcal{T}_{19}$$

Conclusions:

- Abstract framework for ILG for non-conforming methods
- Non-conforming VEM for ILG

Further work:

- Can we guarantee convergence of the iterations of the ILG algorithm?
- If so, can we show linear, or similar, convergence?