Numerical Solution of ODEs

Exercise Class

31st October 2025

Embedded RK Methods

Using the *Butcher Tableau*

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_3
\end{array}
$$

we can define a one step method as

$$
\begin{aligned}
\kappa_i &= f\left(t + \tau c_i, x + \tau \sum_{j=1}^{s} a_{ij}\kappa_j\right), \qquad i = 1, \dots, s, \\
\psi(t+\tau, t, x) &= x + \tau \sum_{i=1}^{s} b_i \kappa_i.
\end{aligned}
$$

`ode23` requires two methods:

"low order" method explicit RK $s = 2$

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & 1/2 & 1/2
\end{array}
\qquad
\begin{aligned}
\kappa_1 &= f(t, x), \\
\kappa_2 &= f(t + \tau, x + \tau\kappa_1), \\
\psi(t+\tau, t, x) &= x + 1/2\tau\kappa_1 + 1/2\tau\kappa_2.
\end{aligned}
$$

"high order" method explicit RK $s = 3$

$$
\begin{array}{c|ccc}
0 & & & \\
c_1 & a_{21} & & \\
c_1 & a_{31} & a_{32} & \\
\hline
 & b_1 & b_2 & b_3
\end{array}
\qquad
\begin{aligned}
\overline{\kappa}_1 &= f(t, x), \\
\overline{\kappa}_2 &= f(t + c_1\tau, x + a_{21}\tau\overline{\kappa}_1), \\
\overline{\kappa}_3 &= f(t + c_2\tau, x + a_{31}\tau\overline{\kappa}_1 + a_{32}\tau\overline{\kappa}_2), \\
\psi(t+\tau, t, x) &= x + b_1\tau\overline{\kappa}_1 + b_2\tau\overline{\kappa}_2 + b_3\tau\overline{\kappa}_3.
\end{aligned}
$$

Therefore, the *low order* method requires two evaluations of f, and the *high order* method requires three evaluations of f; hence, five evaluations of f in total.

Note that $\kappa_1 = \overline{\kappa}_2$; therefore, it reduces the number of evaluations of f by one. If we select $c_1 = 1$ and $c_2 = 1/2$ then we have that,

$$
\begin{array}{c|ccc}
0 & & & \\
1 & 1 & & \\
1/2 & 1/4 & 1/4 & \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
\qquad
\begin{aligned}
\overline{\kappa}_1 &= f(t, x) = \kappa_1, \\
\overline{\kappa}_2 &= f(t + \tau, x + \tau\overline{\kappa}_1) = \kappa_2, \\
\overline{\kappa}_3 &= f(t + \tfrac{1}{2}\tau, x + \tfrac{1}{4}\tau\kappa_1 + \tfrac{1}{4}\tau\kappa_2), \\
\psi(t + \tau, t, x) &= x + \tfrac{1}{6}\tau\kappa_1 + \tfrac{1}{6}\tau\kappa_2 + \tfrac{2}{3}\tau\overline{\kappa}_3.
\end{aligned}
$$

Now only need to evaluate κ_1, κ_2, and $\overline{\kappa}_3$; therefore, only need three evaluations of f, which is the same number of evaluations as for just the *high order* method.

We can define the *low order* method as embedded in the *high order* method:

$$
\begin{array}{c|ccc}
0 & & & \\
1 & 1 & & \\
1/2 & 1/4 & 1/4 & \\
\hline
 & 1/2 & 1/2 & \\
 & 1/6 & 1/6 & 2/3
\end{array}
$$

"low order"

"high order"

Exercises

1. Modify `ode23_orig.m` to use the following low and high order methods:

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & 1/2 & 1/2
\end{array}
\qquad
\begin{array}{c|ccc}
0 & & & \\
1/3 & 1/3 & & \\
2/3 & 0 & 2/3 & \\
\hline
 & 1/4 & 0 & 3/4
\end{array}
$$

2. Modify `ode23_orig.m` to use the embedded methods:

$$
\begin{array}{c|ccc}
0 & & & \\
1/2 & 1/2 & & \\
1 & -1 & 2 & \\
\hline
 & 0 & 1 & \\
 & 1/6 & 2/3 & 1/6
\end{array}
$$

3. Compare results and computation time (using `tic` and `toc`) for the two methods generated in the previous questions on various equations (`linsystem`, `logistic`, `oscillator`)

4. Study `gauss2.m`