

Numerical Solution of ODEs

Exercise Class

31st October 2025

Embedded RK Methods

Using the *Butcher Tableau*

c_1	a_{11}	a_{12}	$\cdots$	a_{1s}
c_2	a_{21}	a_{22}	$\cdots$	a_{2s}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
c_s	a_{s1}	a_{s2}	$\cdots$	a_{ss}
	b_1	b_2	$\cdots$	b_s

we can define a one step method as

$$\kappa_i = f \left(t + \tau c_i, x + \tau \sum_{j=1}^s a_{ij} \kappa_j \right), \quad i = 1, \dots, s,$$

$$\psi(t + \tau, t, x) = x + \tau \sum_{i=1}^s b_i \kappa_i.$$

ode23 requires two methods:

“low order” method explicit RK $s = 2$

0		$\kappa_1 = f(t, x),$
1	1	$\kappa_2 = f(t + \tau, x + \tau \kappa_1),$
	$1/2 \quad 1/2$	$\psi(t + \tau, t, x) = x + 1/2 \tau \kappa_1 + 1/2 \tau \kappa_2.$

“high order” method explicit RK $s = 3$

0			$\bar{\kappa}_1 = f(t, x),$
c_1	a_{21}		$\bar{\kappa}_2 = f(t + c_1 \tau, x + a_{21} \tau \bar{\kappa}_1),$
c_1	a_{31}	a_{32}	$\bar{\kappa}_3 = f(t + c_2 \tau, x + a_{31} \tau \bar{\kappa}_1 + a_{32} \tau \bar{\kappa}_2),$
	b_1	$b_2 \quad b_3$	$\psi(t + \tau, t, x) = x + b_1 \tau \bar{\kappa}_1 + b_2 \tau \bar{\kappa}_2 + b_3 \tau \bar{\kappa}_3.$

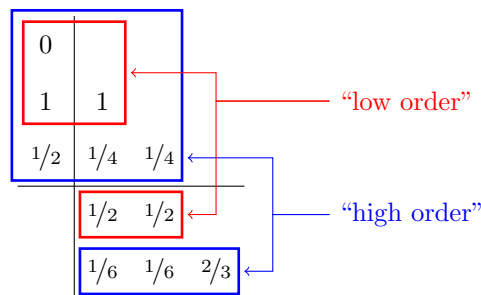
Therefore, the *low order* method requires two evaluations of f , and the *high order* method requires three evaluations of f ; hence, five evaluations of f in total.

Note that $\kappa_1 = \bar{\kappa}_2$; therefore, it reduces the number of evaluations of f by one. If we select $c_1 = 1$ and $c_2 = 1/2$ then we have that,

0		$\bar{\kappa}_1 = f(t, x) = \kappa_1,$
1	1	$\bar{\kappa}_2 = f(t + \tau, x + \tau \bar{\kappa}_1) = \kappa_2,$
$1/2$	$1/4 \quad 1/4$	$\bar{\kappa}_3 = f(t + \frac{1}{2}\tau, x + \frac{1}{4}\tau \kappa_1 + \frac{1}{4}\tau \kappa_2),$
$1/6$	$1/6 \quad 2/3$	$\psi(t + \tau, t, x) = x + \frac{1}{6}\tau \kappa_1 + \frac{1}{6}\tau \kappa_2 + \frac{2}{3}\tau \bar{\kappa}_3.$

Now only need to evaluate κ_1 , κ_2 , and $\bar{\kappa}_3$; therefore, only need three evaluations of f , which is the same number of evaluations as for just the *high order* method.

We can define the *low order* method as embedded in the *high order* method:



Exercises

1. Modify `ode23_orig.m` to use the following low and high order methods:

0		0	
1	1	$1/3$	$1/3$
$1/2$	$1/2$	$2/3$	$0 \quad 2/3$
$1/6$	$1/6$	$1/4$	$0 \quad 3/4$

2. Modify `ode23_orig.m` to use the embedded methods:

0	
$1/2$	$1/2$
1	$-1 \quad 2$
$1/6$	$0 \quad 1$
$1/6$	$1/6 \quad 2/3 \quad 1/6$

3. Compare results and computation time (using `tic` and `toc`) for the two methods generated in the previous questions on various equations (`linsystem`, `logistic`, `oscillator`)
4. Study `gauss2.m`