

Numerical Solution of ODEs

Exercise Class

24th October 2025

Convergence Analysis

Theorem 2.23. *Let there exist a positive constant C such that the local discretisation error is bounded by*

$$d(t + \tau, t, u(t)) \leq C\tau^{p+1},$$

for all $\tau \leq \tau_1$, $t \in [t_0, T]$. Consider an equidistant partition $\{t_j\}_{j=0}^N$ and approximate solution $\{u_j\}_{j=0}^N$, where

$$u_0 = x_0, \quad u_{j+1} = \psi(t_{j+1}, t_j, u_j), \quad j = 0, \dots, N-1.$$

Then,

$$\|u(t_j) - u_j\| \leq \frac{e^{\Lambda(t_j - t_0)} - 1}{\Lambda} C \tau^p, \quad j = 1, \dots, N.$$

Here, p is the order of the method.

If we study the error at the last time step

$$\underbrace{\|u(T) - u_N\|}_{\mathcal{E}_N} \leq \underbrace{\frac{e^{\Lambda(T - t_0)} - 1}{\Lambda} C}_{K = \text{constant}} \tau^p;$$

then,

$$\log_{10} \mathcal{E}_N \leq \log_{10} K + p \log_{10} \tau.$$

Hence, we should observe asymptotically as $\tau \rightarrow 0$ that

$$\log_{10} \mathcal{E}_N = q + p \log_{10} \tau,$$

where $q = \log_{10} K$ is a constant.

Adaptive Timestepping

Algorithm 2.1. Given *two* one step methods

ψ — Order p

$\bar{\psi}$ — Order $p + 1$.

we define adaptive timestepping at each timestep as:

```
 $\tau \leftarrow \max(\tau, \text{tol})$   
 $\delta \leftarrow \|\bar{\psi}(t + \tau, t, x) - \psi(t + \tau, t, x)\|$   
while  $\delta > \text{tol}$  do  
     $\tau \leftarrow \tau \left(\frac{\text{tol}}{\delta}\right)^{1/(p+1)}$   
     $\delta \leftarrow \|\bar{\psi}(t + \tau, t, x) - \psi(t + \tau, t, x)\|$   
end while
```

Accept τ $\triangleright$ it now holds that $\|\bar{\psi}(t + \tau, t, x) - \psi(t + \tau, t, x)\| \leq \text{tol}$
 $t \leftarrow t + \tau$
 $x \leftarrow \bar{\psi}(t + \tau, t, x)$

We provide three implementations of an algorithm with $p = 1$:

ode12_1.m Basic algorithm (using Euler and Heun)

ode12_2.m Adds damping to the timestep size update to prevent large changes.

ode12.m Adds heuristics for the initial timestep size.

Exercises

1. Modify `one_step_order.m` to calculate the order of the Runge, Runge-Kutta, Heun, Implicit Euler, and Crank-Nicholson methods, using the logistic equation

$$\begin{aligned} x'(t) &= (a - bx(t))x(t), & t \in [0, 2], \\ x(0) &= x_0, \end{aligned}$$

with $a = b = 1$, $x_0 = 2$, and known exact solution

$$x(t) = \frac{x_0 e^t}{1 - x_0(1 - e^t)}.$$

Additionally, for implicit Euler and Crank-Nicholson, try with different tolerances in the fixed point iteration.

Remark. Note that the implicit Euler method `ieuler` may not converge for $\tau = 1/2$. Therefore, the convergence analysis code needs to be changed to start from $\tau = 1/4$.

2. Modify `ode12` to use Euler (order $p = 1$ method) and Runge (order $p + 1 = 2$ method). Solve

$$x'(t) = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} x(t), \quad t \in [0, 0.1], \quad (1)$$

$$x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (2)$$

implemented by `linsystem.m`, using this method.