Finite Element Methods 1

Homework 2

Due date: 17th November 2025

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on 17th November 2025.

1. (2 points) Let T be an n-simplex, let $\{a_i\}_{i=1}^n$, $\{a_{iij}\}_{i\neq j}$, $\{a_{ijk}\}_{i<j<k}$ be the points of $L_3(T)$ and let $\{p_i\}_{i=1}^n$, $\{p_{iij}\}_{i\neq j}$, $\{p_{ijk}\}_{i<j<k}$ be the corresponding basis functions of $P_3(T)$. For $i<j<k$ define the linear functionals

$$\Phi_{ijk}(p) = 12\,p(a_{ijk}) + 2\sum_{\ell\in\{i,j,k\}} p(a_\ell) - 3\sum_{\substack{\ell,m\in\{i,j,k\}\\ \ell\neq m}} p(a_{llm})$$

and the space

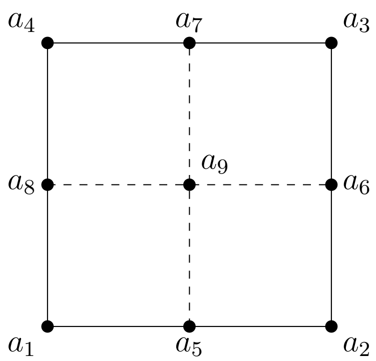
$$P_3'(T) = \{p \in P_3(T) : \Psi_{ijk}(p) = 0, 1 \le i < j < k \le n+1\}.$$

Prove that any function from the space $P_3'(T)$ is uniquely determined by its values at the points $\{a_i\}_{i=1}^n \cup \{a_{iij}\}_{i\neq j}$ and derive basis functions such that each basis function equals 1 at one of these points and vanishes at the rest.

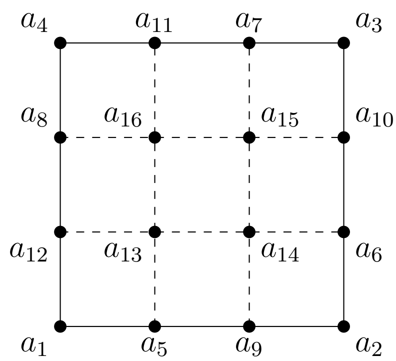
Hint. The basis functions $\{p_i\}_{i=1}^n$, $\{p_{iij}\}_{i\neq j}$ can be modified by adding linear combinations of the functions $\{p_{ijk}\}_{i<j<k}$ in such a way that the resulting functions are in $P_3'(T)$. Show that these function form a basis of $P_3'(T)$ and find formulas for these basis functions.

2. (2 points) Consider finite elements (T, P_T, Σ_T), where

$$\begin{aligned} &T \text{ is a rectangle,}\\ &P_T = Q_3(T),\\ &\Sigma_T = \{p(z) : z \in M_3(T)\}. \end{aligned}$$



(a) $M_2(T)$



(b) $M_3(T)$

Figure 1: Principal lattices for rectangles

For $T = [0,1]^2$, and the points from the principal lattice $M_3(T)$ numbered as per Figure 1b, write basis functions of the finite element (T, P_T, Σ_T). It is sufficient to derive functions for only four basis functions, as the remaining twelve can be obtained by circular permutations of the indices. Let $\mathcal{T}_h$ be a triangulation of a bounded domain $\Omega \subset \mathbb{R}^2$ consisting of rectangles and assign the above finite element to each $T \in \mathcal{T}_h$. Write the definition of the corresponding finite element space X_h and verify that $X_h \subset C(\overline{\Omega})$.

3. (2 points) Let T be a pentahedral prism, see Figure 2, with vertices $a_1, \dots, a_6$. The triangular faces are orthogonal to the x_3 axis, and the quadrilateral faces are parallel to the x_3 axis. Let

$$P_T = \{p(x_1, x_2, x_3) = \gamma_1 + \gamma_2 x_1 + \gamma_3 x_2 + \gamma_4 x_3 + \gamma_5 x_1 x_3 + \gamma_6 x_2 x_3 : \gamma_1, \dots, \gamma_6 \in \mathbb{R}\}.$$

Show that any function $p \in P_T$ is uniquely determined by its values at the vertices $a_1, \dots, a_6$ and that, for any $p \in P_T$ and face $F \subset \partial T$, the restriction $p|_F$ is uniquely determined by its values at the vertices of the face F.

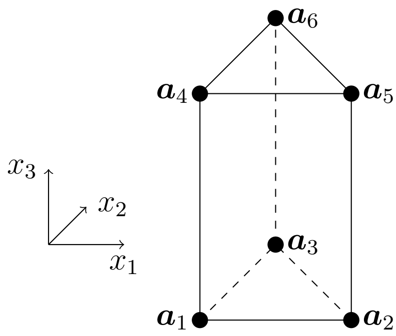


Figure 2: Pentahedral prism