Finite Element Methods 1

Homework 1

Due date: 3rd November 2025

Submit a PDF/scan of the answers to the following questions before the deadline via the *Study Group Roster* (*Záznamník učitele*) in SIS, or hand-in directly at the practical class on 3rd November 2025.

1. (2 points) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz continuous boundary and let $\Gamma \subset \partial\Omega$ be a subset of the boundary of Ω with positive surface measure. For any $p \in [1, \infty)$ prove that there exists a positive constant C such that

$$\|u\|_{0,p,\Omega} \le C \left(|u|_{1,p,\Omega} + \left| \int_\Gamma u \,\mathrm{d}s \right| \right)$$

for all $u \in W^{1,p}(\Omega)$.

2. (2 points) Consider the boundary value problem

$$\begin{aligned} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j} \right) + cu &= f && \text{in } \Omega, \\ \sum_{i,j=1}^{n} n_i a_{ij} \frac{\partial u}{\partial x_j} + hu &= g && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz continuous boundary, $a_{ij} \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $h \in L^\infty(\partial\Omega)$, and $g \in L^2(\partial\Omega)$. We assume the matrix $(a_{ij})_{i,j=1}^n$ is uniformly positive definite a.e. in Ω, $c \ge 0$ a.e. in Ω, and $h \ge h_0$ on $\partial\Omega$ where h_0 is a positive constant.

Derive the variational formulation for the above boundary value problem, using the test space $V = H^1(\Omega)$, and prove a unique solution exists.

3. (2 points) Consider the Poisson equation on the unit square with homogeneous boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (0,1)^2 \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where f is a constant.

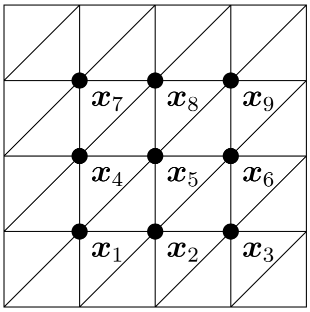
We define the finite element method for this problem as: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle F, v_h \rangle \qquad \text{for all } v_h \in V_h, \tag{3.2}$$

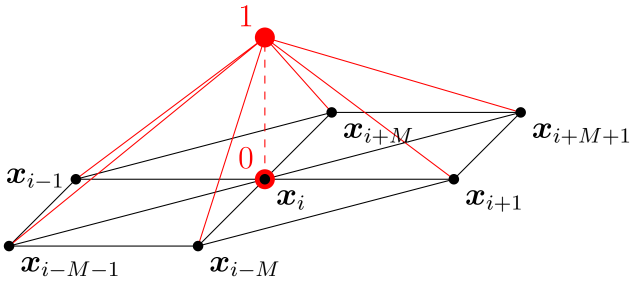
where

$$a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}\boldsymbol{x}, \qquad \langle F, v_h \rangle = \int_\Omega f v_h \,\mathrm{d}\boldsymbol{x},$$

and V_h is finite-dimensional subspace of $H_0^1(\Omega)$. Let $\varphi_1, \dots, \varphi_N$ be the basis functions of V_h; then, the solution u_h of the finite element discretization (3.2) can be



(a) Example of 4×4 triangular mesh



(b) Nodal linear basis function φ_i

Figure 1: Question 2

written in the form $u_h = \sum_{j=1}^{N} u_j \varphi_j$. Hence, the discretization (3.2) is equivalent to solving the following linear system of N unknown coefficients $u_1, \ldots, u_N$:

$$\sum_{j=1}^{N} a(\varphi_j, \varphi_i) u_j = \langle F, \varphi_i \rangle \qquad \text{for } i = 1, \ldots, N. \tag{3.3}$$

We denote by $\mathcal{T}_h$ the triangulation of Ω into triangles in the following manner:

1. subdivide the domain into $(M+1) \times (M+1)$ squares of equal size,
2. divide each square into two triangles by splitting from the bottom left to top-right corner of the square;

see Figure 1a for an example when $M = 3$. We define the width and height of each square as $h = {}^{1}/_{(M+1)}$. Let

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in P_1(T) \ \forall T \in \mathcal{T}_h\};$$

i.e. the space of continuous piecewise linear functions vanishing on the boundary of Ω. To the interior vertices $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ of $\mathcal{T}_h$, where $N = M^2$, (see Figure 1a for one possible numbering of the vertices) we assign a basis function of V_h such that

$$\varphi_i(x_j) = \delta_{ij} \qquad \text{for } i, j = 1, \ldots, N.$$

The support of the basis function φ_i consists of the six triangles sharing the vertex $\boldsymbol{x}_i$, see Figure 1b. This implies that every row of the matrix for the linear system (3.3) contains at most seven non-zero entries.

Compute the entries for the matrix and right-hand side vector for the linear system (3.3) and compare these entries to a discretization using the finite difference scheme on a uniform square mesh.

Hint. Computation of these entries is fairly trivial. Consider, for example, the calculation of $a(\varphi_j, \varphi_i)$, where $j = i + 1$. The nodes $\boldsymbol{x}_j$ and $\boldsymbol{x}_{i+1}$ are connected by an edge and only two triangles share this edge; see Figure 1b. We denote these two triangles as T_1 and T_2, and note that $\operatorname{supp} \varphi_j \cap \operatorname{supp} \varphi_i = T_1 \cup T_2$. Note, also, that $\nabla \varphi_j$ and $\nabla \varphi_i$ are constant on each triangle; therefore,

$$a(\varphi_j, \varphi_i) = \int_{T_1 \cup T_2} \nabla \varphi_j \cdot \nabla \varphi_i \, \mathrm{d}\boldsymbol{x} = |T_1| \, (\nabla \varphi_j)|_{T_1} \cdot (\nabla \varphi_i)|_{T_1} + |T_2| \, (\nabla \varphi_j)|_{T_2} \cdot (\nabla \varphi_i)|_{T_2}.$$

The derivatives of φ_j and φ_i with respect to x and y can be computed on the horizontal and vertical edges, respectively, of the triangles T_1 and T_2.