Numerical Solution of ODEs

Exercise Class

12th December 2024

Domains of Stability (Multistep)

For m-step methods,

$$\begin{aligned} &a_m u_{j+m} + a_{m-1} u_{j+m-1} + \cdots + a_0 u_j \\ &\qquad = \tau \left(b_m f(t_{j+m}, u_{j+m}) + b_{m-1} f(t_{j+m-1}, u_{j+m-1}) + \cdots + b_0 f(t_j, u_j) \right), \end{aligned}$$

we have the domain of stability

$$S = \{\mu \in \mathbb{C} : \forall z \in \mathbb{C}, \rho(z) - \mu\sigma(z) = 0 \in \mathbb{C} \implies |z| < 1\}$$

where

$$\rho(z) = \sum_{i=0}^{m} a_i z^i, \qquad \text{and} \qquad \sigma(z) = \sum_{i=0}^{m} b_i z^i.$$

We can explicitly define the boundary of S as the curve

$$\mu = \frac{\rho(\mathrm{e}^{i\theta})}{\sigma(\mathrm{e}^{i\theta})}$$

for $\theta \in [0, 2\pi)$.

Stiff problems — Semi-discretisation of PDEs

Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} && \text{in } \Omega \coloneqq \{(t,x), 0 \le x \le \pi, t_0 \le t \le T\}, \\ u(0,t) = u(\pi,t) &= 0, && \text{for } t \in [t_0, T], \\ u(x,0) &= u^0(x), && \text{for } x \in [0, \pi]. \end{aligned}$$

Via the method of lines and a *central difference* approximation at discrete spatial points $x_j = hj$, $j = 1, \ldots, N$, $h = \frac{\pi}{N+1}$ of

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{h^2}, \qquad j = 1, \ldots, N,$$

we derive the following initial value:

$$\boldsymbol{u}'(t) = A\boldsymbol{u}(t) \qquad \text{for } t \in [t_0, T],$$

with initial conditions $\boldsymbol{u}^0 \in \mathbb{R}^N$ at time $t_0 = 0$, given by $u_j^0 = u^0(x_j)$, $j = 1, \ldots, N$, where

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}.$$

This is the Jacobian of the initial value problem at any point.

Exercises

1. Use `mstep_stab.m` to plot the boundary of the domains of stability for BDF1, BDF2, BDF3, and BDF4.

   Note that the domain of stability is the *exterior* of this boundary.

2. Solve this system for $N = 100, 1000$.

3. Numerically compute (using MATLAB) the spectrum $\sigma(A)$ and stiffness ratio

$$L = \frac{\max_{i=1,\ldots,N} |\mathrm{Re}(\lambda_i)|}{\min_{i=1,\ldots,N} |\mathrm{Re}(\lambda_i)|}$$

   for $N = 10, 100, 1000$.