Numerical Solution of ODEs

Exercise Class

28th November 2024

Domains of Stability

For the explicit Runge-Kutta methods, we have the following domains of stability:

$$
\begin{aligned}
&s = 1 \text{ (Euler)}: && S = \{\mu \in \mathbb{C} : |1+\mu| < 1\} \\
&s = 2 \text{ (Runge)}: && S = \left\{\mu \in \mathbb{C} : \left|1+\mu+\frac{1}{2}\mu^2\right| < 1\right\} \\
&s = 3: && S = \left\{\mu \in \mathbb{C} : \left|1+\mu+\frac{1}{2}\mu+\frac{1}{6}\mu^3\right| < 1\right\} \\
&s = 4: && S = \left\{\mu \in \mathbb{C} : \left|1+\mu+\frac{1}{2}\mu+\frac{1}{6}\mu^3+\frac{1}{24}\mu^4\right| < 1\right\}
\end{aligned}
$$

Exercises

1. Given the pendulum with dynamical friction:

$$
\begin{aligned}
x_1' &= x_2, \\
x_2' &= -k \sin x_1 + \varepsilon x_2,
\end{aligned}
$$

consider the linearised version around the steady state $(0,0)$:

$$
x' = Ax, \qquad A = \begin{pmatrix} 0 & 1 \\ -k & \varepsilon \end{pmatrix}
$$

with initial conditions $x_0 \in \mathbb{R}^2$ at time $t_0 = 0$, setting $k = 1$ and $\varepsilon = -0.7$.

Using MATLAB plot the eigenvalues of the matrix A and determine for $\tau = 0.1, 0.2, 0.4, 1$ whether the fixed point $(0,0)$ is A-stable for the first, second, third, and fourth order Runge-Kutta methods.

2. Consider the linear ODE

$$
x' = Ax, \qquad A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -2 & 1 \end{pmatrix}
$$

Determine whether the steady state $(0,0)$ is A-stable and estimate (by plotting $\tau\lambda$ against the domain of stability) values of the step size τ which will result in an A-stable fixed point for the first, second, third, and fourth order Runge-Kutta methods.