2-step Nyström

$$u_{j+1} = u_{j-1} + \sum_{i=0}^{1} f_{j+i-1} \int_{t_{j-1}}^{t_{j+1}} \prod_{\substack{k=0 \\ k\neq i}}^{1} \frac{s - t_{j+k-1}}{t_{j+i-1} - t_{j+k-1}}\, ds$$

$$= u_{j-1} + f_{j-1} \int_{t_{j-1}}^{t_{j+1}} \frac{s - t_j}{t_{j-1} - t_j}\, ds + f_j \int_{t_{j-1}}^{t_{j+1}} \frac{s - t_{j-1}}{t_j - t_{j-1}}\, ds$$

$$= u_{j-1} + f_{j-1} \left[\frac{\frac{1}{2}s^2 - t_j s}{t_{j-1} - t_j} \right]_{t_{j-1}}^{t_{j+1}} + f_j \left[\frac{\frac{1}{2}s^2 - t_{j-1} s}{t_{j-1} - t_j} \right]_{t_{j-1}}^{t_{j+1}}$$

$$= u_{j-1} + \frac{1}{\tau} f_{j-1} \left(\tfrac{1}{2} t_{j+1}^2 - t_j t_{j+1} - \tfrac{1}{2} t_{j-1}^2 - t_j t_{j-1} \right) + \frac{1}{\tau} f_j \left(\tfrac{1}{2} t_{j+1}^2 - t_{j-1} t_{j+1} - \tfrac{1}{2} t_{j-1}^2 + t_{j-1}^2 \right)$$

$$= u_{j-1} + \frac{1}{\tau} f_{j-1} \left(\tfrac{1}{2} (t_{j+1} - t_{j-1})(t_{j+1} + t_{j-1}) - t_j (t_{j+1} - t_{j-1}) \right) + \frac{1}{\tau} f_j \left(\tfrac{1}{2} (t_{j+1} - t_{j-1})(t_{j+1} + t_{j-1}) - t_{j-1} (t_{j+1} - t_{j-1}) \right)$$

$$= u_{j-1} + \frac{1}{2\tau} f_{j-1} \left(\tau (t_{j+1} + t_{j-1}) - 2\tau t_j \right) + \frac{1}{2\tau} f_j \left(\tau (t_{j+1} + t_{j-1}) - 2\tau t_{j-1} \right)$$

$$= u_{j-1} + f_{j-1} \left(\underbrace{\tfrac{t_{j+1} - t_j}{2}}_{\tau} + \underbrace{\tfrac{t_{j-1} - t_j}{2}}_{-\tau} \right) + f_j \underbrace{\left(\tfrac{t_{j+1} - t_{j-1}}{2} \right)}_{2\tau}$$

$$= u_{j-1} + 2\tau f_j \equiv \text{1-step Nyström.}$$

2-step Milne-Simpson

$$u_{j+1} \approx u_{j-1} + \sum_{i=0}^{2} f_{j-1+i} \int_{t_{j-1}}^{t_{j+1}} \prod_{\substack{k=0 \\ k\neq i}}^{2} \frac{s - t_{j-1+k}}{t_{j-1+i} - t_{j-1+k}}\, ds$$

$$= u_{j-1} + f_{j-1} \int_{t_{j-1}}^{t_{j+1}} \left(\frac{s - t_j}{t_{j-1} - t_j} \right) \left(\frac{s - t_{j+1}}{t_{j-1} - t_{j+1}} \right) ds + f_j \int_{t_{j-1}}^{t_{j+1}} \left(\frac{s - t_{j-1}}{t_j - t_{j-1}} \right) \left(\frac{s - t_{j+1}}{t_j - t_{j+1}} \right) ds + f_j \int_{t_{j-1}}^{t_{j+1}} \left(\frac{s - t_{j-1}}{t_{j+1} - t_{j-1}} \right) \left(\frac{s - t_j}{t_{j+1} - t_j} \right) ds$$

$$= u_{j-1} + f_{j-1}\frac{1}{2\tau^2}\left[\frac{1}{3}\left(t_{j+1}^3 - t_{j-1}^3\right) - \frac{1}{2}\left(t_j + t_{j+1}\right)\left(t_{j+1}^2 - t_{j-1}^2\right) + t_j t_{j+1}\left(t_{j+1} - t_{j-1}\right)\right]$$
$$+ f_j \frac{1}{\tau^2}\left[\frac{1}{3}\left(t_{j+1}^3 - t_{j-1}^3\right) - \frac{1}{2}\left(t_{j-1} + t_{j+1}\right)\left(t_{j+1}^2 - t_{j-1}^2\right) + t_{j-1} t_{j+1}\left(t_{j+1} - t_{j-1}\right)\right]$$
$$+ f_{j+1}\frac{1}{2\tau^2}\left[\frac{1}{3}\left(t_{j+1}^3 - t_{j-1}^3\right) - \frac{1}{2}\left(t_{j-1} + t_j\right)\left(t_{j+1}^2 - t_{j-1}^2\right) + t_{j-1} t_j\left(t_{j+1} - t_{j-1}\right)\right]$$

$$\left(t_{j+1} - t_{j-1}\right)^3 = t_{j+1}^3 - 3t_{j+1}^2 t_{j-1} + 3t_{j+1} t_{j-1}^2 - t_{j-1}^3$$
$$\Rightarrow t_{j+1}^3 - t_{j-1}^3 = \underbrace{\left(t_{j+1} - t_{j-1}\right)^3}_{8\tau^3} + \underbrace{3t_{j+1}^2 t_{j-1} - 3t_{j+1} t_{j-1}^2}_{3t_{j+1} t_{j-1}\left(t_{j+1} - t_{j-1}\right)}$$

$$u_{j+1} = u_{j-1} + f_{j-1}\frac{1}{2\tau^2}\left[\frac{8}{3}\tau^3 + 2t_{j+1}\left(t_{j-1} + t_j\right)\tau - \tau\left(t_j t_{j+1} + t_{j+1}^2 + t_j t_{j-1} + t_{j+1} t_{j-1}\right)\right]$$
$$+ f_j \frac{1}{\tau^2}\left[\frac{8}{3}\tau^3 + 4t_{j+1} t_{j-1}\tau - \tau\left(t_{j-1}^2 + 2t_{j+1} t_{j-1} + t_{j+1}^2\right)\right]$$
$$+ f_{j+1}\frac{1}{2\tau^2}\left[\frac{8}{3}\tau^3 + 2t_{j-1}\left(t_{j+1} + t_j\right)\tau - \tau\left(t_j t_{j+1} + t_j t_{j+1} + t_{j-1} t_j + t_{j-1}^2\right)\right]$$

$$= u_{j-1} + f_{j-1}\frac{1}{2\tau}\left[\frac{8}{3}\tau^2 + \underbrace{\left(t_j - t_{j+1}\right)}_{-\tau}\underbrace{\left(t_{j+1} - t_{j-1}\right)}_{2\tau}\right]$$
$$+ f_j \frac{1}{\tau}\left[\frac{8}{3}\tau^2 - \underbrace{\left(t_{j-1} - t_{j+1}\right)^2}_{(-2\tau)^2}\right]$$
$$+ f_{j+1}\frac{1}{2\tau}\left[\frac{8}{3}\tau^2 + \underbrace{\left(t_{j-1} - t_j\right)}_{-\tau}\underbrace{\left(t_{j+1} - t_{j-1}\right)}_{2\tau}\right]$$

$$= u_{j-1} + \tau\left(\frac{1}{3}f_{j-1} + \frac{4}{3}f_j + \frac{1}{3}f_{j+1}\right)$$