Numerical Solution of ODEs

Exercise Class

14th November 2024

Multistep — Numerical Integration Definition

For $q, k \in \mathbb{N}_0$ and $\ell \in \{0, 1\}$

$$u_{j+1} = u_{j-k} \approx \sum_{i=0}^{q+\ell} f_{j-q+i} \underbrace{\int_{t_{j-k}}^{t_{j+1}} \mathcal{L}_{j-q+i}(s)\,\mathrm{d}s}_{b},$$

where

$$\mathcal{L}_{j-q+i}(s) = \prod_{\substack{k=0\\k\neq i}}^{q+\ell} \frac{s - t_{j-q+k}}{t_{j-q+i} - t_{j-q+k}}, \qquad t_{j-q} \leq s \leq t_{j+1}, i = 0, \ldots, q+\ell.$$

Predictor/Corrector

For implicit methods we have previously used either a fixed-point or Newton method to solve. Alternatively, we can use the *Predictor/Corrector* method, where we combine a implicit and explicit method:

Predictor An explicit method (e.g. Adams-Bashfort 2)

Corrector An implicit method (e.g. Adams-Moulton 2)

Algorithm 3.2 ($PECE$)**.** For the time step t_{j+1}:

Predict (ab2): $u_{j+1}^P = u_j + \tau\left(\frac{3}{2}f(t_j, u_j) - \frac{1}{2}f(t_{j-1}, u_{j-1})\right)$

Evaluate: $f_{j+1}^E = f(t_{j+1}, u_{j+1}^P)$

Correct (am2): $u_{j+1}^C = u_j + \tau\left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j, u_j) - \frac{1}{12}f(t_{j-1}, u_{j-1})\right)$

Evaluate: $f_{j+1}^E = f(t_{j+1}, u_{j+1}^C)$

Define $u_{j+1} = u_{j+1}^C$, $f(t_{j+1}, u_{j+1}) = f_{j+1}^E$.

Algorithm 3.3 (PEC)**.** For the time step t_{j+1}:

Predict (ab2): $u_{j+1}^P = u_j + \tau\left(\frac{3}{2}f(t_j, u_j) - \frac{1}{2}f(t_{j-1}, u_{j-1})\right)$

Evaluate: $f_{j+1}^E = f(t_{j+1}, u_{j+1}^P)$

Correct (am2): $u_{j+1}^C = u_j + \tau\left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j, u_j) - \frac{1}{12}f(t_{j-1}, u_{j-1})\right)$

Define $u_{j+1} = u_{j+1}^C$, $f(t_{j+1}, u_{j+1}) = f_{j+1}^E$.

Algorithm 3.4 ($PECECE = P(EC)^2E$)**.** For the time step t_{j+1}:

$$\begin{aligned}
&\text{Predict (ab2):} && u_{j+1}^P = u_j + \tau\left(\frac{3}{2}f(t_j,u_j) - \frac{1}{2}f(t_{j-1},u_{j-1})\right)\\
&\text{Evaluate:} && f_{j+1}^E = f(t_{j+1},u_{j+1}^P)\\
&\text{Correct (am2):} && u_{j+1}^C = u_j + \tau\left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j,u_j) - \frac{1}{12}f(t_{j-1},u_{j-1})\right)\\
&\text{Evaluate:} && f_{j+1}^E = f(t_{j+1},u_{j+1}^C)\\
&\text{Correct (am2):} && u_{j+1}^C = u_j + \tau\left(\frac{5}{11}f_{j+1}^E + \frac{2}{3}f(t_j,u_j) - \frac{1}{12}f(t_{j-1},u_{j-1})\right)\\
&\text{Evaluate:} && f_{j+1}^E = f(t_{j+1},u_{j+1}^C)
\end{aligned}$$

Define $u_{j+1} = u_{j+1}^C$, $f(t_{j+1},u_{j+1}) = f_{j+1}^E$.

By repeating the Evaluate/Correct steps k-times, $k \in \mathbb{N}$, we can derive the algorithms $P(EC)^kC$ and $P(EC)^k$.

BDF (Backward Differentiation Formula) Methods

This is an m-step method of the highest-order where $b_0 = \cdots = b_{m-1} = 0$ and $a_m = 1$. We derive by solving a linear system:

$$\begin{aligned}
\text{BDF1}:&\quad u_{j+1} - u_j = \tau f(t_{j+1},u_{j+1}),\\
\text{BDF2}:&\quad u_{j+2} - \frac{4}{3}u_{j+1} + \frac{1}{3}u_j = \frac{2}{3}\tau f(t_{j+2},u_{j+2}),\\
\text{BDF3}:&\quad u_{j+3} - \frac{18}{11}u_{j+2} + \frac{9}{11}u_{j+1} - \frac{2}{11}u_j = \frac{6}{11}\tau f(t_{j+3},u_{j+3}),\\
\text{BDF4}:&\quad u_{j+4} - \frac{48}{25}u_{j+3} + \frac{36}{25}u_{j+2} - \frac{16}{25}u_{j+1} + \frac{3}{25}u_j = \frac{12}{25}\tau f(t_{j+4},u_{j+4}),\\
\text{BDF5}:&\quad u_{j+5} - \frac{300}{137}u_{j+4} + \frac{300}{137}u_{j+3} - \frac{200}{137}u_{j+2} + \frac{75}{137}u_{j+1} - \frac{12}{137}u_j = \frac{60}{137}\tau f(t_{j+5},u_{j+5}),\\
\text{BDF6}:&\quad u_{j+6} - \frac{360}{147}u_{j+5} + \frac{450}{137}u_{j+4} - \frac{400}{147}u_{j+3} + \frac{225}{147}u_{j+2}\\
&\qquad - \frac{72}{147}u_{j+1} + \frac{10}{147}u_j = \frac{60}{147}\tau f(t_{j+6},u_{j+6}),
\end{aligned}$$

m-step BDF is order $p = m$. Note that BDF1, ..., BDF6 are D-stable, but for $m \geq 7$ BDF is *not* D-stable.

Exercises

1. Derive the recursive formula for the following methods using the numerical integration definition:
   (a) 2-step Nyström method ($k = 1$, $\ell = 0$, $q = 1$)
   (b) 2-step Milne-Simpson method ($k = 1$, $\ell = 1$, $q = 1$)
2. Compare the numerical solutions given by the 2-step Milne-Simpson, Nyström, and Adams-Moulton methods, along with the numerical solution from `ode23` for the following ODEs.

(a) Linear oscillator (`oscillator.m`) on the time interval $t \in [0, 10]$, with $\tau = 0.1$:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -9x + 10\cos(t), \\ \boldsymbol{x}(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

For comparisons plot t vs. x_1.

(b) Stiff linear system (`linsystem.m`) on the time interval $t \in [0, 0.05]$ with $\tau = 0.001$:

$$\begin{aligned} \boldsymbol{x}' &= \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}(0) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

Also run convergence analysis using `conv_analysis.m` to deduce the order of the *Milne-Simpson*, and *Nyström* 2-step methods.

3. Create a method to implement BDF4, and compare the numerical solutions given by BDF2, BDF3, BDF4, 2-step Adams-Moulton and `ode23` for the ODEs from Question 2.

   Also run convergence analysis using `conv_analysis.m` to deduce the order of BDF2, BDF3, and BDF4.