

Numerical Solution of ODEs

Exercise Class

31st October 2024

Embedded RK Methods

Using the *Butcher Tableau*

$$\begin{array}{c|cccc}
 c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
 c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
 \hline
 & b_1 & b_2 & \cdots & b_s
 \end{array}$$

we can define a one step method as

$$\kappa_i = f \left(t + \tau c_i, x + \tau \sum_{j=1}^s a_{ij} \kappa_j \right), \quad i = 1, \dots, s,$$

$$\psi(t + \tau, t, x) = x + \tau \sum_{i=1}^s b_i \kappa_i.$$

ode23 requires two methods:

“low order” method explicit RK $s = 2$

$$\begin{array}{c|cc}
 0 & & \\
 1 & 1 & \\
 \hline
 & 1/2 & 1/2
 \end{array}
 \quad
 \begin{aligned}
 \kappa_1 &= f(t, x), \\
 \kappa_2 &= f(t + \tau, x + \tau \kappa_1), \\
 \psi(t + \tau, t, x) &= x + 1/2 \tau \kappa_1 + 1/2 \tau \kappa_2.
 \end{aligned}$$

“high order” method explicit RK $s = 3$

$$\begin{array}{c|ccc}
 0 & & & \\
 c_1 & a_{21} & & \\
 c_1 & a_{31} & a_{32} & \\
 \hline
 & b_1 & b_2 & b_3
 \end{array}
 \quad
 \begin{aligned}
 \bar{\kappa}_1 &= f(t, x), \\
 \bar{\kappa}_2 &= f(t + c_1 \tau, x + a_{21} \tau \bar{\kappa}_1), \\
 \bar{\kappa}_3 &= f(t + c_2 \tau, x + a_{31} \tau \bar{\kappa}_1 + a_{32} \tau \bar{\kappa}_2), \\
 \psi(t + \tau, t, x) &= x + b_1 \tau \bar{\kappa}_1 + b_2 \tau \bar{\kappa}_2 + b_3 \tau \bar{\kappa}_3.
 \end{aligned}$$

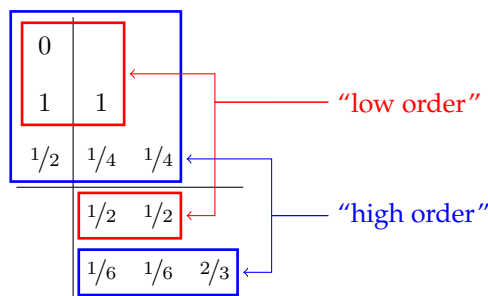
Therefore, the *low order* method requires two evaluations of f , and the *high order* method requires three evaluations of f ; hence, five evaluations of f in total.

Note that $\kappa_1 = \bar{\kappa}_2$; therefore, it reduces the number of evaluations of f by one. If we select $c_1 = 1$ and $c_2 = 1/2$ then we have that,

0			$\bar{\kappa}_1 = f(t, x) = \kappa_1,$
1	1		$\bar{\kappa}_2 = f(t + \tau, x + \tau\bar{\kappa}_1) = \kappa_2,$
1/2	1/4	1/4	$\bar{\kappa}_3 = f(t + \frac{1}{2}\tau, x + \frac{1}{4}\tau\kappa_1 + \frac{1}{4}\tau\kappa_2),$
	1/6	1/6	$\psi(t + \tau, t, x) = x + \frac{1}{6}\tau\kappa_1 + \frac{1}{6}\tau\kappa_2 + \frac{2}{3}\tau\bar{\kappa}_3.$

Now only need to evaluate $\kappa_1, \kappa_2,$ and $\bar{\kappa}_3$; therefore, only need three evaluations of f , which is the same number of evaluations as for just the *high order* method.

We can define the *low order* method as embedded in the *high order* method:



Exercises

1. Modify `ode23_orig.m` to use the following low and high order methods:

0			0		
1	1		1/3	1/3	
	1/2	1/2	2/3	0	2/3
			1/4	0	3/4

2. Modify `ode23_orig.m` to use the embedded methods:

0			
1/2	1/2		
1	-1	2	
	0	1	
	1/6	2/3	1/6

3. Compare results and computation time (using `tic` and `toc`) for the two methods generated in the previous questions on various equations (`linsystem`, `logistic`, `oscillator`)
4. Study `gauss2.m`