Nonlinear Differential Equations

Practical 11: Linearisation & Iterative Methods

1. Consider the following boundary value boundary value problem in the bounded Lipschitz domain $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$: For $2 \leq p < \infty$, $f \in L^q(\Omega)$, $1/p + 1/q = 1$,

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = f, \qquad \text{in } \Omega,$$
$$u = 0, \qquad \text{on } \partial\Omega.$$

(a) State a linearised, iterative, version of this equation (Kačanov method)

Solution: Given an initial guess $u^{(0)}$, solve for $m = 0, 1, \dots$

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u^{(m)}}{\partial x_i} \right|^{p-2} \frac{\partial u^{(m+1)}}{\partial x_i} \right) = f, \qquad \text{in } \Omega,$$
$$u = 0, \qquad \text{on } \partial\Omega.$$

(b) State the weak formulation of both the nonlinear and linearised iterative method

Solution: For $u, v, w \in W^{1,p}(\Omega)$ define

$$a(u; v, w) = \int_\Omega \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}\boldsymbol{x}$$
$$F(v) = \int_\Omega f v \, \mathrm{d}\boldsymbol{x};$$

then, the weak formulation are as follows:

Nonlinear : Find $u \in W^{1,p}(\Omega)$ such that

$$a(u; u, v) = \ell(v) \qquad \text{for all } v \in W^{1,p}(\Omega)$$

Linear : Given an initial guess $u^{(0)}$, for $m = 0, 1, \dots$ find $u^{(m+1)} \in W^{1,p}(\Omega)$ such that

$$a(u^{(m)}; u^{(m+1)}, v) = \ell(v) \qquad \text{for all } v \in W^{1,p}(\Omega).$$

(c) State the Galerkin formulation of both the nonlinear and linearised iterative method

Solution: Define a finite dimensional subspace $X_n \subset W^{1,p}(\Omega)$; then, the Galerkin approximations are as follows:

Nonlinear : Find $u_n \in X_n$ such that

$$a(u_h; u_h, v) = \ell(v) \qquad \text{for all } v \in X_n$$

> **Linear** : Given an initial guess $u_h^{(0)}$, for $m = 0, 1, \dots$ find $u_h^{(m+1)} \in X_n$ such that
> $$a(u_h^{(m)}; u_h^{(m+1)}, v) = \ell(v) \qquad \text{for all } v \in X_n.$$

2. Consider the following boundary value boundary value problem in the bounded Lipschitz domain $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$

$$\begin{aligned} -\varepsilon\Delta u &= f(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

with $\varepsilon > 0$ and a damped Newton iteration approximation: Find $u^{(m+1)} \in H^1(\Omega)$ such that

$$a_\varepsilon(u^{(m)}; u^{(m+1)}, u^{(m)}) = a_\varepsilon(u^{(m)}; u^{(m)}, u^{(m)}) - \varepsilon_m \ell_\varepsilon(u^{(m)}; v) \qquad \forall v \in H^1(\Omega)$$

where $\varepsilon_m \in (0, 1]$ and

$$\begin{aligned} a_\varepsilon(u; w, v) &= \int_\Omega (\varepsilon \nabla w \cdot \nabla v - f'(u) w v) \,\mathrm{d}\boldsymbol{x}, \\ \ell_\varepsilon(u; v) &= \int_\Omega (\varepsilon \nabla u \cdot \nabla v - f(u) v) \,\mathrm{d}\boldsymbol{x}, \end{aligned}$$

Define the norm

$$|||u|||^2 = \varepsilon \|\nabla u\|_{0,2}^2 + \|u\|_{0,2}^2;$$

then; if there exists positive constants $\underline{\lambda}, \overline{\lambda}$ with $\varepsilon C_P^{-2} > \overline{\lambda}$, such that $-\underline{\lambda} \le f'(u) \le \overline{\lambda}$ for all $u \in \mathbb{R}$, where C_P is the Poincáre constant from the Poincáre inequality

$$\|w\|_{0,2} \le C_P \|\nabla w\|_{0,2}.$$

show that, for fixed $u \in X$

(a) $a_\varepsilon(u; \cdot, \cdot)$ is bounded; i.e., there exists a positive constant $\alpha > 0$ (depending on u) such that

$$a_\varepsilon(u; w, v) \le \alpha |||w||| \, |||v||| \qquad \text{for all } v, w \in H^1(\Omega),$$

> **Solution:** We first note that as $-\underline{\lambda} \le f'(u) \le \overline{\lambda}$ then $|f'(u)| \le \max(\underline{\lambda}, \overline{\lambda})$ for all $u \in \mathbb{R}$; then.
> $$\begin{aligned} a_\varepsilon(u; w, v) &\le \int_\Omega |\varepsilon \nabla w \cdot \nabla v| + \int_\Omega |f'(u)||wv| \,\mathrm{d}\boldsymbol{x} \\ &\le \varepsilon \|\nabla w\|_{0,2} \|\nabla v\|_{0,2} + \max(\underline{\lambda}, \overline{\lambda}) \|w\|_{0,2} \|v\|_{0,2} \\ &\le \max(1, \underline{\lambda}, \overline{\lambda}) |||w||| \, |||v|||. \end{aligned}$$

(b) $a_\varepsilon(u; v, v)$ is coercive; i.e., there exists a positive constant $\beta > 0$ (depending on u) such that

$$a_\varepsilon(u; v, v) \ge \beta |||v|||^2 \qquad \text{for all } v \in H^1(\Omega),$$

Solution: As $f'(u) \leq \overline{\lambda}$; then, $-f'(u) \leq -\overline{\lambda}$ and by the Poincáre inequality

$$\begin{aligned} a_\varepsilon(u; v, v) &= \int_\Omega \varepsilon |\nabla v|^2 - \int_\Omega f'(u)|v|^2 \, \mathrm{d}\boldsymbol{x} \\ &\geq \varepsilon \|\nabla v\|_{0,2}^2 - \overline{\lambda}\|v\|_{0,2}^2 \\ &\geq \frac{\varepsilon}{C_P^2}\|v\|_{0,2}^2 - \overline{\lambda}\|v\|_{0,2}^2 \\ &= \left(\frac{\varepsilon}{C_P^2} - \overline{\lambda}\right)\|v\|_{0,2}^2 \end{aligned}$$

Similarly,

$$a_\varepsilon(u; v, v) \geq \left(\varepsilon - \overline{\lambda} C_P^2\right)\|\nabla v\|_{0,2}^2.$$

Combining these results,

$$\left(\frac{\varepsilon}{C_P^2} + 1\right) a_\varepsilon(u; v, v) \geq \frac{\varepsilon\left(\varepsilon - \overline{\lambda} C_P^2\right)}{C_P^2}\|\nabla v\|_{0,2}^2 + \left(\frac{\varepsilon}{C_P^2} - \overline{\lambda}\right)\|v\|_{0,2}^2 = \left(\frac{\varepsilon}{C_P^2} - \overline{\lambda}\right)|\!|\!| v |\!|\!|^2$$

As $\varepsilon C_P^{-2} > \overline{\lambda}$ then setting

$$\beta = \frac{\varepsilon C_P^{-2} - \overline{\lambda}}{\varepsilon C_P^{-2} + 1} > 0$$

completes the proof.

(c) $a_\varepsilon(u; u, \cdot) - \varepsilon_m \ell_\varepsilon(u; \cdot)$ is bounded; i.e., there exists a positive constant $\gamma > 0$ (depending on u) such that

$$a_\varepsilon(u; u, v) - \varepsilon_m \ell_\varepsilon(u; v) \leq \gamma |\!|\!| v |\!|\!| \qquad \text{for all } v \in H^1(\Omega).$$

Solution:

$$\begin{aligned} &a_\varepsilon(u; u, v) - \varepsilon_m \ell_\varepsilon(u; v) \\ &\qquad = \int_\Omega \left(\varepsilon \nabla u \cdot \nabla v - f'(u)uv\right) \mathrm{d}\boldsymbol{x} - \varepsilon_m \int_\Omega \left(\varepsilon \nabla u \cdot \nabla v - f(u)v\right) \mathrm{d}\boldsymbol{x} \\ &\qquad \leq \varepsilon|1 - \varepsilon_m|\|\nabla u\|_{0,2}\|\nabla v\|_{0,2} + \left(\min(\underline{\lambda}, \overline{\lambda})\|u\|_{0,2} + \varepsilon_m\|f(u)\|_{0,2}\right)\|v\|_{0,2} \end{aligned}$$

As $u \in H^1(\Omega)$ then $\|\nabla u\|_{0,2}$ and $\|u\|_{0,2}$ are bounded. Assume also that $\|f(u)\|_{0,2}$ is bounded; then the proof is complete.