Nonlinear Differential Equations

Practical 11: Linearisation & Iterative Methods

1. Consider the following boundary value boundary value problem in the bounded Lipschitz domain $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$: For $2 \leq p < \infty$, $f \in L^q(\Omega)$, $1/p + 1/q = 1$,

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right) = f, \qquad \text{in } \Omega,$$
$$u = 0, \qquad \text{on } \partial\Omega.$$

   (a) State a linearised, iterative, version of this equation (Kačanov method)
   (b) State the weak formulation of both the nonlinear and linearised iterative method
   (c) State the Galerkin formulation of both the nonlinear and linearised iterative method

2. Consider the following boundary value boundary value problem in the bounded Lipschitz domain $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$

$$-\varepsilon\Delta u = f(u) \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

   with $\varepsilon > 0$ and a damped Newton iteration approximation: Find $u^{(m+1)} \in H^1(\Omega)$ such that

$$a_\varepsilon(u^{(m)}; u^{(m+1)}, u^{(m)}) = a_\varepsilon(u^{(m)}; u^{(m)}, u^{(m)}) - \varepsilon_m \ell_\varepsilon(u^{(m)}; v) \qquad \forall v \in H^1(\Omega)$$

   where $\varepsilon_m \in (0, 1]$ and

$$a_\varepsilon(u; w, v) = \int_\Omega (\varepsilon\nabla w \cdot \nabla v - f'(u)wv)\,\mathrm{d}\boldsymbol{x},$$
$$\ell_\varepsilon(u; v) = \int_\Omega (\varepsilon\nabla u \cdot \nabla v - f(u)v)\,\mathrm{d}\boldsymbol{x},$$

   Define the norm

$$|||u|||^2 = \varepsilon\|\nabla u\|_{0,2}^2 + \|u\|_{0,2}^2;$$

   then; if there exists positive constants $\underline{\lambda}, \overline{\lambda}$ with $\varepsilon C_P^{-2} > \overline{\lambda}$, such that $-\underline{\lambda} \leq f'(u) \leq \overline{\lambda}$ for all $u \in \mathbb{R}$, where C_P is the Poincáre constant from the Poincáre inequality

$$\|w\|_{0,2} \leq C_P\|\nabla w\|_{0,2}.$$

   show that, for fixed $u \in X$

   (a) $a_\varepsilon(u; \cdot, \cdot)$ is bounded; i.e., there exists a positive constant $\alpha > 0$ (depending on u) such that

$$a_\varepsilon(u; w, v) \leq \alpha|||w|||\,|||v||| \qquad \text{for all } v, w \in H^1(\Omega),$$

   (b) $a_\varepsilon(u; v, v)$ is coercive; i.e., there exists a positive constant $\beta > 0$ (depending on u) such that

$$a_\varepsilon(u; v, v) \geq \beta|||v|||^2 \qquad \text{for all } v \in H^1(\Omega),$$

   (c) $a_\varepsilon(u; u, \cdot) - \varepsilon_m\ell_\varepsilon(u; \cdot)$ is bounded; i.e., there exists a positive constant $\gamma > 0$ (depending on u) such that

$$a_\varepsilon(u; u, v) - \varepsilon_m\ell_\varepsilon(u; v) \leq \gamma|||v||| \qquad \text{for all } v \in H^1(\Omega).$$