Nonlinear Differential Equations

Practical 10: Potential Operators

1. Show that every monotone and potential operator is demicontinuous (Lemma 3.18).

   Hint. *Use Lemma 3.14, Theorem 3.16, and Lemma 3.17.*

Solution: Let $A : X \to X'$ be monotone; then, by Lemma 3.14 to show A is demicontinuous it is sufficient to show that

$$\langle f - Aw, u - w\rangle \geq 0 \text{ for all } w \in X \qquad \Longrightarrow \qquad Au = f, f \in X'. \tag{1.1}$$

Let $w = u + t(v - u) \in X$, for al $v \in X$; then,

$$\begin{aligned}
-\langle f, t(v-u)\rangle + \langle Aw, t(u-v)\rangle &\geq 0 \\
\langle f, t(v-u)\rangle &\leq t\langle Aw, v-u\rangle \\
&= t\langle Aw, v - w + t(v-u)\rangle \\
&= t\langle Aw, (v + t(v-u)) - w\rangle.
\end{aligned} \tag{1.2}$$

If A is a potential operator there exists a potential F with Gâteaux derivative $F' \equiv A$. By Theorem 3.16

$$A \text{ monotone} \qquad \Longleftrightarrow \qquad F(y) \geq F(x) + \langle Ax, y - x\rangle \quad \forall x, y \in X;$$

therefore, letting $x = w$, $y = v + t(v-u)$ and applying to (1.2)

$$\langle f, t(v-u)\rangle \leq t\langle Aw, (v + (t(v-u)) - w)\rangle \leq t(F(v + t(v-u)) - F(w)).$$

Dividing by $t \neq 0$ and take the limit as $t \to 0$ gives

$$\begin{aligned}
&\langle f, v-u\rangle \leq \lim_{t\to 0}(F(v + t(v-u)) - F(u + t(v-u))) = F(v) - F(u) \\
\Longrightarrow \qquad & F(u) - \langle f, u\rangle \leq F(v) - \langle f, v\rangle \qquad \forall v \in X \\
\Longrightarrow \qquad & F(u) - \langle f, u\rangle = \min_{v\in X}(F(v) - \langle f, v\rangle).
\end{aligned}$$

By Lemma 3.16 we have that $Au = f$, $f \in X'$ has a solution; hence, we have shown (1.1) holds. Therefore, A is demicontinuous.

2. Let $A : X \to X'$ be a monotone potential operator. Then, show that $u \in X$ is a solution of $Au = f$, $f \in X'$, if and only if

$$\int_0^1 \langle Atu, u\rangle \, \mathrm{d}t - \langle f, u\rangle = \min_{v\in X}\left[\int_0^1 \langle Atv, v\rangle \, \mathrm{d}t - \langle f, v\rangle\right].$$

   (Lemma 3.19)

Solution: As A is a monotone potential operator; then A is demicontinuous by Lemma 3.18 which is equivalent to A being radially continuous by Lemma 3.14. From Lemma 3.15

$$F(v) = F(0) + \int_0^1 \langle Atv, v\rangle \, \mathrm{d}t, \qquad v \in X.$$

By Lemma 3.17 there exists a solution to solution to $Au = f$, for any $f \in X'$ if and only if

$$\begin{aligned}
& & F(u) - \langle f, u\rangle &= \min_{v\in X}(F(v) - \langle f, v\rangle) \\
&\iff & F(0) + \int_0^1 \langle Atu, u\rangle \, \mathrm{d}t - \langle f, u\rangle &= \min_{v\in X}\left(F(0) + \int_0^1 \langle Atv, v\rangle \, \mathrm{d}t - \langle f, v\rangle\right) \\
&\iff & \int_0^1 \langle Atu, u\rangle \, \mathrm{d}t - \langle f, u\rangle &= \min_{v\in X}\left(\int_0^1 \langle Atv, v\rangle \, \mathrm{d}t - \langle f, v\rangle\right).
\end{aligned}$$

3. Let $A : X \to X'$ be a strictly monotone, coercive, potential operator with potential F. For any $f \in X'$ prove that there exists a unique solution $u \in X$ of $Au = f$ which minimises the potential of the problem $G = F - f$ and that

$$\begin{aligned}
G(u) &\equiv F(u) - \langle f, u\rangle \\
&= \min_{v\in X}\left(\int_0^1 \langle Atv, v\rangle \, \mathrm{d}t - \langle f, v\rangle\right) \\
&= -\int_0^1 \langle f, A^{-1}tf\rangle \, \mathrm{d}t + \int_0^1 \langle AtA^{-1}0, A^{-1}0\rangle \, \mathrm{d}t.
\end{aligned}$$

(Corollary 3.25)

Solution: As A is a strictly monotone coercive potential operator $Au = f$ has a unique solution $u \in X$ by Theorem 3.20. Now, it is necessary to show that this solution minimises G:

$$\begin{aligned}
G(u) \equiv F(u) - \langle f, u\rangle &= \int_0^1 \langle Atu, u\rangle \, \mathrm{d}t - \langle f, u\rangle && \text{(Theorem 3.24)} \\
&= \min_{v\in X}\left(\int_0^1 \langle Atv, v\rangle \, \mathrm{d}t - \langle f, v\rangle\right) && \text{(Lemma 3.19)} \\
&= \min_{v\in X}\left(F(v) - \langle f, v\rangle\right) && \text{(Theorem 3.24)} \\
&= \min_{v\in X} G(v)
\end{aligned}$$

Finally, by applying the various equalities in Theorem 3.24:

$$\begin{aligned}
G(u) \equiv F(u) - \langle f, u\rangle = F(u) - \langle Au, u\rangle &= -F^*(Au) = -F^*(f) \\
&= -F^*(0) - \int_0^1 \langle f, A^{-1}tf\rangle \, \mathrm{d}t \\
&= F(A^{-1}0) - \int_0^1 \langle f, A^{-1}tf\rangle \, \mathrm{d}t \\
&= \int_0^1 \langle AtA^{-1}0, A^{-1}0\rangle \, \mathrm{d}t - \int_0^1 \langle f, A^{-1}tf\rangle \, \mathrm{d}t.
\end{aligned}$$