Nonlinear Differential Equations

Practical 10: Potential Operators

1. Show that every monotone and potential operator is demicontinuous (Lemma 3.18).

   Hint. *Use Lemma 3.14, Theorem 3.16, and Lemma 3.17.*

2. Let $A : X \to X'$ be a monotone potential operator. Then, show that $u \in X$ is a solution of $Au = f$, $f \in X'$, if and only if

   $$\int_0^1 \langle Atu, u \rangle \, \mathrm{d}t - \langle f, u \rangle = \min_{v \in X} \left[\int_0^1 \langle Atv, v \rangle \, \mathrm{d}t - \langle f, v \rangle \right].$$

   (Lemma 3.19)

3. Let $A : X \to X'$ be a strictly monotone, coercive, potential operator with potential F. For any $f \in X'$ prove that there exists a unique solution $u \in X$ of $Au = f$ which minimises the potential of the problem $G = F - f$ and that

   $$\begin{aligned} G(u) &\equiv F(u) - \langle f, u \rangle \\ &= \min_{v \in X} \left(\int_0^1 \langle Atv, v \rangle \, \mathrm{d}t - \langle f, v \rangle \right) \\ &= - \int_0^1 \langle f, A^{-1} tf \rangle \, \mathrm{d}t + \int_0^1 \langle AtA^{-1}0, A^{-1}0 \rangle \, \mathrm{d}t. \end{aligned}$$

   (Corollary 3.25)