Nonlinear Differential Equations

Practical 9: Brouwer Fixed Point

1. Prove Theorem 3.6 from the lecture:

   Theorem 3.6. *Let X be a finite-dimensional normed linear space $K \subset X$ a closed, convex, and bounded subset, and $f : K \to K$ a continuous mapping. Then, there exists a fixed point of f in K; i.e., $\exists \overline{x} \in K$ such that*

   $$\overline{x} = f(\overline{x}).$$

   Hint. *Define*

   $$x = \sum_{i=1}^{n} \alpha_i x_i,$$

   where $x_1, \ldots, x_n$ form a basis for X ($\dim X = n$), and $\boldsymbol{\alpha} = \{\alpha_i\}_i^n \in \mathbb{R}^n$. Then, define a linear, continuous operator $T : X \to \mathbb{R}^n$ as $T(x) = \boldsymbol{\alpha}$ (which has a continuous inverse T^{-1}). Defining $K_1 = T(K)$ and $g(\boldsymbol{\alpha}) = T \circ f \circ T^{-1}\boldsymbol{\alpha}$, show that T is a homeomorphism and g has a fixed point, and hence, that f has a fixed point.

Solution: Using the definition of T from the hint, let $K_1 = T(K)$. We need to show that K_1 is convex, bounded, and maps K_1 to itself. We want to show that K_1 is a convex, closed, and bounded subset of $\mathbb{R}^n$. In order to do that, we first show T is a homeomorphism:

T **and** T^{-1} **continuous** by definition

T **is surjective:** $T : K \to K_1$, so clearly true to definition

T **is injective:** Assume there exists $x, y \in K$, where

$$x = \sum_{i=1}^{n} \alpha_i x_i, \qquad y = \sum_{i=1}^{n} \beta_i x_i$$

with $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$. If $T(x) = T(y)$; then,

$$\begin{aligned} & \boldsymbol{\alpha} = \boldsymbol{\beta} \\ \implies \quad & \alpha_i = \beta_i, \qquad i = 1, \ldots, n \\ \implies \quad & x = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \beta_i x_i = y. \end{aligned}$$

Therefore, $T(x) = T(y) \implies x = y$.

Now, as T is a homeomorphism then T is a closed map and, hence, it maps the closed set K to the closed set K_1. Furthermore, we can sow K_1 is convex. For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in K_1$, there exists a $x, y \in K$, such that $\boldsymbol{\alpha} = T(x)$ and $\boldsymbol{\alpha} = T(y)$. As K is convex, for $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda) y \in K \qquad \implies \qquad T(\lambda x + (1 - \lambda) y) \in K_1.$$

As
$$x = \sum_{i=1}^{n} \alpha_i x_i, \qquad y = \sum_{i=1}^{n} \beta_i x_i;$$
then,
$$\begin{aligned} \lambda x + (1-\lambda) y &= \sum_{i=1}^{n} (\lambda \alpha_i + (1-\lambda)\beta_i) x_i \\ \implies \qquad T(\lambda x + (1-\lambda) y) &= (\lambda \alpha_i + (1-\lambda)\beta_i)_{i=1}^{n} \\ &= \lambda \boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta}. \end{aligned}$$

Hence, $\lambda \boldsymbol{\alpha} + (1-\lambda)\boldsymbol{\beta} \in K_1$ and K_1 is convex.

We now have a continuous function $g(\boldsymbol{\alpha}) = T \circ f \circ T^{-1}(\boldsymbol{\alpha})$ (T, T^{-1}, and f are all continuous) which we can show maps from K_1 to K_1:

$$\begin{array}{rrll} & \forall \boldsymbol{\alpha} \in K_1 \quad \exists x \in K & \text{such that} & T^{-1}(\boldsymbol{\alpha}) = x, \\ & \forall x \in K \quad \exists y \in K & \text{such that} & f(x) = y, \\ & \forall y \in K \quad \exists \boldsymbol{\beta} \in K_1 & \text{such that} & T(y) = \beta, \\ \implies & \forall \boldsymbol{\alpha} \in K_1 \quad \exists \boldsymbol{\beta} \in K_1 & \text{such that} & g(\boldsymbol{\alpha}) = T \circ f \circ T^{-1}(\boldsymbol{\alpha}) = \beta. \end{array}$$

Therefore, for all $\boldsymbol{\alpha} \in K_1$, $g(\boldsymbol{\alpha}) \in K_1$; i.e., $g : K_1 \to K_1$.

So, $g : K_1 \to K_1$ is a continuous functional on a closed, convex, bounded set $K_1 \subset \mathbb{R}^n$; hence, by Theorem 3.5 there exists a fixed point $\overline{\boldsymbol{\alpha}}$ such that
$$g(\overline{\boldsymbol{\alpha}}) = \overline{\boldsymbol{\alpha}};$$
hence, there exists a $\overline{x} \in K$ such that
$$\overline{x} = T^{-1}(\overline{\boldsymbol{\alpha}}) \qquad \implies \qquad \overline{\boldsymbol{\alpha}} = T(\overline{x}).$$
Then, as T is bijective we have that
$$\begin{aligned} g(\overline{\boldsymbol{\alpha}}) &= \overline{\boldsymbol{\alpha}} \\ T \circ f \circ T^{-1}(\overline{\boldsymbol{\alpha}}) &= T(\overline{x}) \\ T \circ f(\overline{x}) &= T(\overline{x}) \\ f(\overline{x}) &= \overline{x}. \end{aligned}$$
Therefore, $\overline{x}$ is a fixed point of f.

2. Let the conditions of Theorem 2.11/Corollary 3.8 be met; i.e., $A : X \to X'$ monotone, coercive, and hemicontinuous on a real *separable* reflexive Banach space X.

   (a) Show that if A is strictly monotone that the inverse A^{-1} exists and is strictly monotone, demicontinuous, and bounded.

   Hint. *For demicontinuous, let $v_n = A^{-1} f_n$, $f_n \to f$. Show sequence $\{v_n\}$ is bounded; hence, there exists a subsequence $v_{n'} \rightharpoonup v$, and show $v = A^{-1} f$. Then, show holds for whole sequence (see Proposition 1.8).*

Solution: In the proof we showed that for a strictly monotone operator $Au = f$ has a unique solution for all $f \in X'$; i.e., A is injective and surjective (bijective). This implies that $A^{-1} : X' \to X$ exists.
Then, let $Au_1 = f_1$, $Au_2 = f_2$, with $f_1 \neq f_2 \implies u_1 \neq u_2$, we have that

$$\langle f_1 - f_2, A^{-1}f_1 - A^{-1}f_2 \rangle = \langle Au_1 - Au_2, u_1 - u_2 \rangle > 0;$$

by the fact that A is strictly monotone; hence A^{-1} is strictly monotone.
Boundedness of A^{-1} follows from coercivity!
Let $v_n = A^{-1}f_n$, $f_n \to f$; then, $\{v_n\}$ is bounded due to the boundedness of A^{-1}; hence, exists a subsequence $v_{n'} \rightharpoonup v$. Additionally,

$$\langle f - Aw, v - w \rangle = \lim_{n \to \infty} f_{n'} - Aw, v_{n'} - w \geq 0$$

for all $w \in X$. As A hemicontinuous it follows that $Av = f \implies v = A^{-1}f$. So $v_{n'} \rightharpoonup A^{-1}f$ and by Proposition 1.8 $A^{-1}f_n \rightharpoonup A^{-1}f$; hence demicontinuity is proven.

(b) If A is uniformly monotone, show that A^{-1} is continuous.

Solution: As A is uniformly monotone, there exists a $a : \mathbb{R} \to \mathbb{R}$ which is strictly increasing with $a(0) = 0$ such that

$$a(\|u - v\|)\|u - v\| \leq \langle Au - Av, u - v \rangle \leq \|Au - Av\|\|u - v\|;$$

hence,

$$a(\|u - v\|) \leq \|Au - Av\|.$$

Then, we have with $Au = f_n$, $Av = f$ that

$$a(\|A^{-1}f_n - A^{-1}f\|) \leq \|f_n - f\|.$$

Hence, if $f_n \to tof$; then

$$a(\|A^{-1}f_n - A^{-1}f\|) \to 0 \qquad \implies \qquad \|A^{-1}f_n - A^{-1}f\| \to 0,$$

due to properties of a; therefore, $A^{-1}f_n \to A^{-1}f$

(c) If A is strongly monotone, show that A^{-1} is Lipschitz continuous.

Solution: If A is strongly monotone; then, from the above with $a(\|u - v\|) = M\|u - v\|$ we have with $Au = f_1$, $Av = f_2$ that

$$M\|u - v\| \leq \|Au - Av\| \quad \implies \quad \|A^{-1}f_1 - A^{-1}f_2\| \leq \frac{1}{M}\|f_1 - f_2\| \;\; \forall f_1, f_2 \in X'.$$

Hence, A^{-1} Lipschitz continuous with constant $1/M$.

3. Let $A : X \to X'$ be a bounded operator on a real, separable, reflexive, and infinite-dimensional Banach space X and $f \in X'$. Let $\{v_1, v_2, \dots\}$ be the basis of X and there

exists a $R > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\langle Au_n - f, v_k \rangle = 0, \qquad u_n \in X_n, \qquad k = 1, \dots, n, \tag{3.1}$$

where $X_n = \operatorname{span}\{v_1, \dots, v_n\}$ has a solution u_n with $\|u_n\| \leq R$.

(a) If A satisfies (M), show that there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ with $u_{n'} \rightharpoonup u$ such that $u \in X$ is a solution of $Au = f$.

Hint. *Consider the limit as $n \to \infty$ of* (3.1) *and show the left hand side of* (M) *is satisfied.*

Solution: From the Galerkin approximation we have that $\langle Au_n - f, v \rangle \to 0$ for all $v \in \operatorname{span}\{v_1, v_2, \dots\}$. As A is bounded the sequence $\{Au_n\}$ is bounded; therefore, $Au_n \rightharpoonup f$ in X'. As $\{u_n\}$ is bounded in a reflexive Banach space there exists a subsequence $u_{n'} \rightharpoonup u$, and from definition of the Galerkin approximation

$$\langle Au_{n'}, u_{n'} \rangle = \langle f, u_{n'} \rangle \to \langle f, u \rangle.$$

Hence, we have that

$$u_{n'} \rightharpoonup u, \quad Au_{n'} \rightharpoonup f, \quad \limsup_{n \to \infty} \langle Au_{n'}, u_{n'} \rangle \leq \langle f, u \rangle \quad \implies \quad Au = f$$

by (M).

(b) If A satisfies $(\mathrm{S})_0$ and is demicontinuous, show that there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ with $u_{n'} \to u$ such that $u \in X$ is a solution of $Au = f$.

Hint. *Show left hand side of* $(\mathrm{S})_0$ *is satisfied.*

Solution: In part (a) we found that

$$u_{n'} \rightharpoonup u, \quad Au_{n'} \rightharpoonup f, \quad \lim_{n \to \infty} \langle Au_{n'}, u_{n'} \rangle = \langle f, u \rangle;$$

hence, by $(\mathrm{S})_0$, $u_{n'} \to u$. As A is demicontinuous, this implies that $Au_{n'} \rightharpoonup Au$; hence, $Au = f$.