Nonlinear Differential Equations

Practical 7: Semimonotone Operators

Let X be a real, separable, reflexive Banach space and $B : X \times X :\to X'$ be a map such that

$$Au = B(u,u) \qquad \text{for all } u \in X.$$

The operator $A : X \to X'$ is called *semimonotone* if and only if the following hold.

a) For all $u, v \in X$
$$\langle B(u,u) - B(u,v), u - v\rangle \geq 0.$$

b) For each $u \in X$, the operator $v \mapsto B(u,v)$ is hemicontinuous and bounded from X to X', and, for each $v \in X$, the operator $u \mapsto B(u,v)$ is hemicontinuous and bounded from X to X'.

c) If $u_n \rightharpoonup u$ in X and
$$\lim_{n\to\infty} \langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle = 0;$$
then, $B(u_n, v) \rightharpoonup B(u,v)$ in X' for all $v \in X$,

d) Let $v \in X$, $u_n \rightharpoonup u$ in X, and $B(u_n,v) \rightharpoonup w$ in X' as $n \to \infty$; then,
$$\lim_{n\to\infty} \langle B(u_n,v), u_n\rangle = \langle w, u\rangle.$$

e) A is bounded.

Exercises

1. Let $A : X \to X'$ be a semimonotone operator on a real, separable, reflexive Banach space X, and $B : X \times X \to X'$ the associated map. Assume that $u_n \rightharpoonup u$, $B(u_n,u) \rightharpoonup w$ and
$$\limsup_{n\to\infty} \langle B(u_n,u_n), u_n - u\rangle \leq 0.$$

   (a) Show that
$$\lim_{n\to\infty} \langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle = 0;$$
   i.e, show that the condition of property c) of a semimonotone operator is satisfied.

Solution: By property a), $B(u_n,u) \rightharpoonup w$, and d)

$$\begin{aligned}
0 &\leq \liminf_{n\to\infty} \langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle \\
&\leq \limsup_{n\to\infty} \langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle \\
&= \limsup_{n\to\infty} \langle B(u_n,u_n), u_n - u\rangle - \limsup_{n\to\infty} \langle B(u_n,u), u_n\rangle + \limsup_{n\to\infty} \langle B(u_n,u), u\rangle \\
&= \limsup_{n\to\infty} \langle B(u_n,u_n), u_n - u\rangle - \langle w, u\rangle + \langle w, u\rangle.
\end{aligned}$$

Then, by $\limsup_{n\to\infty} \langle B(u_n,u_n), u_n - u\rangle \leq 0$,

$$\lim_{n\to\infty} \langle B(u_n,u_n) - B(u_n,u), u_n - u\rangle = 0.$$

(b) Hence, show that

$$\langle B(u,u), u-w\rangle \le \liminf_{n\to\infty}\langle B(u_n,u_n), u_n-u\rangle \qquad \text{for all } w\in X;$$

i.e., A is a pseudo-monotone operator.

Hint. *Similar to question 3 from last week.*

Solution: Let $z = u + t(w-u)$ and, by property a) and c),

$$\begin{aligned}
&\langle B(u_n,u_n) - B(u_n,z), u_n - z\rangle \ge 0\\
&\quad \liminf_{n\to\infty} t\langle B(u_n,u_n), u-w\rangle \ge \liminf_{n\to\infty}(t\langle B(u_n,z), u-w\rangle - \langle B(u_n,u_n), u_n-u\rangle\\
&\qquad\qquad + \langle B(u_n,z), u_n-u\rangle)\\
&\qquad = \liminf_{n\to\infty}(t\langle B(u_n,z), u-w\rangle - \langle B(u_n,u), u_n-u\rangle\\
&\qquad\qquad + \langle B(u_n,z), u_n-u\rangle)
\end{aligned}$$

As property c) holds from the previous question; then, $B(u_n,v)\rightharpoonup B(u,v)$ in X' for all $v\in X$. Therefore, by property d),

$$B(u_n,u)\rightharpoonup B(u,u) \quad\Longrightarrow\quad \lim_{n\to\infty}\langle B(u_n,u),u_n\rangle = \langle B(u,u),u\rangle \tag{1.1}$$

$$B(u_n,z)\rightharpoonup B(u,z) \quad\Longrightarrow\quad \lim_{n\to\infty}\langle B(u_n,z),u_n\rangle = \langle B(u,z),u\rangle \tag{1.2}$$

Hence,

$$\liminf_{n\to\infty}\langle B(u_n,u_n), u-w\rangle \ge \liminf_{n\to\infty}\langle B(u_n,z), u-w\rangle = \langle B(u,z), u-w\rangle.$$

As B is hemicontinuous in the second argument by property b) let $t\to 0$, and by previous question and (1.1) $\lim_{n\to\infty}\langle B(u_n,u_n),u_n\rangle = \lim_{n\to\infty}\langle B(u_n,u_n),u\rangle$; therefore,

$$\langle B(u,u), u-w\rangle \le \liminf_{n\to\infty}\langle B(u_n,u_n), u_n-w\rangle \qquad \text{for all } w\in X.$$

2. Consider a quasilinear PDE of order $2k$, $k\in\mathbb{N}$ of the form

$$\sum_{|\alpha|\le k}(-1)^\alpha\partial^\alpha a_\alpha(\boldsymbol{x},\delta_k u(\boldsymbol{x})) = f(\boldsymbol{x}) \qquad \text{in } \Omega,$$

$$\frac{\partial^i u}{\partial \boldsymbol{n}^i} = 0 \qquad \text{on } \partial\Omega, i = 1,\dots,k-1,$$

where Ω is a bounded Lipschitz domain. Let $a_\alpha : \Omega\times\mathbb{R}^\kappa\to\mathbb{R}$, for each multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le k$, satisfies the Carathéodory condition **(B1)**, growth condition **(B2)**, and coercivity condition **(C2)** from Theorem 2.19, as well as the following:

(I1) The highest order terms are *strictly monotone* with respect to the highest order derivatives; i.e.,

$$\sum_{|\alpha|=k}\left(a_\alpha(\boldsymbol{x},\eta,\xi) - a_\alpha(\boldsymbol{x},\eta,\widehat{\xi})\right)(\xi_\alpha - \widehat{\xi}_\alpha) > 0,$$

for all $\eta \in \mathbb{R}^{\widetilde{\kappa}}$, $\xi, \widehat{\xi} \in \mathbb{R}^{\widetilde{\kappa}-\kappa}$, where

$$\widetilde{\kappa} = \frac{(n+k-1)!}{n!(k-1)!}$$

is the number of multi-indices of length $|\alpha| \leq k-1$.

(I2) The highest order terms are *coercive* with respect to the highest order derivatives; i.e.,

$$\lim_{|\xi|\to\infty} \sup_{\eta\in D} \sum_{|\alpha|=k} \frac{a_\alpha(\boldsymbol{x},\eta,\xi)}{|\xi|+|\xi|^{p-1}} == \infty,$$

for almost all $x \in \Omega$ and bounded sets $D \subset \mathbb{R}^{\widetilde{\kappa}}$.

Let $A : W_0^{k,p}(\Omega) \to W^{-k,q}(\Omega)$, $Au = B(u,u)$, where

$$\langle B(w,u), v\rangle = \int_\Omega \sum_{|\alpha|=k} a_\alpha(\boldsymbol{x}, \delta_{k-1}w(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x}))\partial^\alpha v \,\mathrm{d}\boldsymbol{x} + \int_\Omega \sum_{|\alpha|\leq k-1} a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x}))\partial^\alpha v \,\mathrm{d}\boldsymbol{x}.$$

(a) Show that for all $u, v \in W_0^{k,p}(\Omega)$

$$\langle B(u,u) - B(u,v), u-v\rangle \geq 0;$$

i.e., prove property a) of a semimonotone operator.

Solution: For $u \neq v$, by (I1),

$$\begin{aligned}&\langle B(u,u) - B(u,v), u-v\rangle \\ &\quad = \int_\Omega \sum_{|\alpha|=k} \Big(a_\alpha(\boldsymbol{x}, \delta_{k-1}u(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x})) - a_\alpha(\boldsymbol{x}, \delta_{k-1}u(\boldsymbol{x}), \widehat{\delta}_k v(\boldsymbol{x}))\Big)\, \partial^\alpha(u-v)\,\mathrm{d}\boldsymbol{x} \\ &\quad > 0.\end{aligned}$$

For $u = v$, $\langle B(u,u) - B(u,v), u-v\rangle = 0$ trivially.

(b) Show that for each $u \in W_0^{k,p}(\Omega)$, the operator $v \mapsto B(u,v)$ is hemicontinuous and bounded from $W_0^{k,p}(\Omega)$ to $W^{-k,q}(\Omega)$; i.e., prove the first part of property b) of a semimonotone operator.

Solution: Let $\{t_n\} \in \mathbb{R}$ be a sequence such that $t_n \to 0$; then, as $a_\alpha(\boldsymbol{x}, \eta)$, $|\alpha| \leq k$, is continuous for all $\eta \in \mathbb{R}^\kappa$

$$\begin{aligned}\langle B(w, u+t_n v), z\rangle &= \int_\Omega \sum_{|\alpha|=k} a_\alpha(\boldsymbol{x}, \delta_{k-1}w(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x}) + t_n \widehat{\delta}_k v(\boldsymbol{x}))\partial^\alpha z(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad + \int_\Omega \sum_{|\alpha|\leq k-1} a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x}))\partial^\alpha v\,\mathrm{d}\boldsymbol{x} \\ &\to \int_\Omega \sum_{|\alpha|=k} a_\alpha(\boldsymbol{x}, \delta_{k-1}w(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x}))\partial^\alpha z(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad + \int_\Omega \sum_{|\alpha|\leq k-1} a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x}))\partial^\alpha v\,\mathrm{d}\boldsymbol{x} = \langle B(w,u), z\rangle\end{aligned}$$

Hence, $v \mapsto B(u, v)$ is hemicontinuous. By **(B2)**

$$
\begin{aligned}
|\langle B(w,u), v\rangle| &\leq \int_\Omega \sum_{|\alpha|=k} |a_\alpha(\boldsymbol{x}, \delta_{k-1}w(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x}))||\partial^\alpha v| \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \sum_{|\alpha|\leq k-1} |a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x}))||\partial^\alpha v| \,\mathrm{d}\boldsymbol{x} \\
&\leq C \int_\Omega \sum_{|\alpha|=k} \left(g_\alpha(\boldsymbol{x}) + \sum_{|\beta|\leq k-1} |\partial^\beta w|^{p-1} + \sum_{|\beta|=k} |\partial^\beta u|^{p-1} \right) |\partial^\alpha v| \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \sum_{|\alpha|\leq k-1} \left(g_\alpha(\boldsymbol{x}) + \sum_{|\beta|\leq k} |\partial^\beta w|^{p-1} \right) |\partial^\alpha v| \,\mathrm{d}\boldsymbol{x} \\
&\leq C \left(\sum_{|\alpha|=k} \|g_\alpha\|_{0,q} + \|w\|_{k,p}^{p/q} + |u|_{k,p}^{p/q} \right) \|v\|_{k,p}
\end{aligned}
$$

hence,

$$
\|B(w,u)\|_{-k,q} = \sup_{v \in W_0^{k,p}(\Omega)} \frac{|\langle B(w,u), v\rangle|}{\|v\|_{k,p}} \leq C \left(\sum_{|\alpha|=k} \|g_\alpha\|_{0,q} + \|w\|_{k,p}^{p/q} + |u|_{k,p}^{p/q} \right). \tag{2.1}
$$

As $g_a \in L^q(\Omega)$ and $w, u \in W_0^{k,p}(\Omega)$ this is bounded.

(c) Show that for each $v \in W_0^{k,p}(\Omega)$, the operator $u \mapsto B(u, v)$ is hemicontinuous and bounded from $W_0^{k,p}(\Omega)$ to $W^{-k,q}(\Omega)$; i.e., prove the second part of property b) of a semimonotone operator.

Solution: Let $\{t_n\} \in \mathbb{R}$ be a sequence such that $t_n \to 0$; then, as $a_\alpha(\boldsymbol{x}, \eta)$, $|\alpha| \leq k$, is continuous for all $\eta \in \mathbb{R}^\kappa$,

$$
\begin{aligned}
\langle B(w + t_n v, u), z\rangle &= \int_\Omega \sum_{|\alpha|=k} a_\alpha(\boldsymbol{x}, \delta_{k-1}w(\boldsymbol{x}) + t_n \delta_{k-1} v(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x})) \partial^\alpha z(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \sum_{|\alpha|\leq k-1} a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x}) + t_n \delta_k v(\boldsymbol{x})) \partial^\alpha v \,\mathrm{d}\boldsymbol{x} \\
&\to \int_\Omega \sum_{|\alpha|=k} a_\alpha(\boldsymbol{x}, \delta_{k-1}w(\boldsymbol{x}), \widehat{\delta}_k u(\boldsymbol{x})) \partial^\alpha z(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \sum_{|\alpha|\leq k-1} a_\alpha(\boldsymbol{x}, \delta_k w(\boldsymbol{x})) \partial^\alpha v \,\mathrm{d}\boldsymbol{x} = \langle B(w,u), z\rangle
\end{aligned}
$$

Hence, $u \mapsto B(u, v)$ is hemicontinuous. Boundedness follows directly from (2.1).

(d) Show that A is bounded; i.e., prove property e) of a semimonotone operator.

Solution: From Lemma 2.14.

(e) Show that A is coercive.

Solution: From Lemma 2.18.

(f) Assume properties c) and d) of a semimonotone operator applies for A without proof. Hence, show that a solution $u \in W_0^{k,p}(\Omega)$ of the weak formulation

$$a(u,v) := \int_\Omega \sum_{|\alpha|\leq k} a_\alpha(\boldsymbol{x}, \delta_k u(\boldsymbol{x}))\partial^\alpha v \,\mathrm{d}\boldsymbol{x} = \int_\Omega fv \,\mathrm{d}\boldsymbol{x}, \qquad \text{for all } v \in W_0^{k,p}(\Omega),$$

exists for each right-hand side $f \in L^q(\Omega)$, $1/p + 1/q = 1$.

Solution: As we have shown that $A : X \to X'$, where

$$\langle Au, v\rangle = a(u,v)$$

is semimonotone and coercive; then, by Lemma 2.34 the equation $Au = F$ has a solution $u \in W_0^{k,p}(\Omega)$ for every $F \in W^{-k,q}(\Omega)$. Furthermore, for every $f \in L^q(\Omega)$ we can show that for

$$\langle F, v\rangle = \int_\Omega fv \,\mathrm{d}\boldsymbol{x}$$

$F \in W^{-k,q}(\Omega)$. Therefore, for every $f \in L^q(\Omega)$ there exists a weak solution $u \in W_0^{k,p}(\Omega)$ to the weak formulation.