Nonlinear Differential Equations

Practical 6: Pseudomonotone Operators

1. Let $A : X \to X'$ be a nonlinear operator on a real Banach space X. Show that
   (a) A monotone and hemicontinuous $\implies$ A satisfies the condition (M),

Solution: Assume the left-hand side of the condition (M) holds; i.e.,

$$u_n \rightharpoonup u, \quad Au_n \rightharpoonup b, \quad \limsup_{n\to\infty} \langle Au_n, u_n\rangle \le \langle b, u\rangle.$$

We first note that for all $v \in X$

$$\begin{aligned}\langle b - Av, u - v\rangle &= \langle b, u\rangle - \langle Av, u\rangle - \langle b - Av, v\rangle \\ &\ge \limsup_{n\to\infty} \left(\langle Au_n, u_n\rangle - \langle Av, u_n\rangle - \langle Au_n - Av, v\rangle\right) \\ &= \limsup_{n\to\infty} \langle Au_n - Av, u_n - v\rangle.\end{aligned}$$

By monotonicity

$$\langle Au_n - Av, u_n - v\rangle \ge 0, \qquad \implies \qquad \langle b - Av, u - v\rangle \ge 0 \quad \forall v \in X.$$

Let $v = u - tw$, where $t > 0$; then,

$$0 \le \langle b - Av, u - v\rangle = \langle b - A(u + tw), w\rangle \quad \forall w \in X.$$

As A is hemicontinuous, as $t \to 0$,

$$\langle b - Au, w\rangle \ge 0 \quad \forall w \in X \qquad \implies \qquad \langle b - Au, w\rangle = 0 \quad \forall w \in X,$$

as setting $w = -w$ the condition still holds. Therefore, $b - Au = 0$, which implies $Au = b$; hence, the right-hand side of (M) is shown.

   (b) A uniformly monotone $\implies$ A satisfies the condition $(\mathrm{S})_+$.

Solution: Assume the left-hand side of the condition $(\mathrm{S})_+$ holds; i.e.,

$$u_n \rightharpoonup u, \quad \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle \le 0.$$

Then, by uniform monotonicity,

$$0 \ge \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle \ge \limsup_{n\to\infty} a(\|u_n - u\|)\|u_n - u\| \ge 0.$$

Therefore, by properties of $a(\,\cdot\,)$

$$\limsup_{n\to\infty} a(\|u_n - u\|)\|u_n - u\| = 0 \qquad \iff \qquad \lim_{n\to\infty} \|u_n - u\| = 0;$$

hence, $u_n \to u$ and the right-hand side of $(\mathrm{S})_+$ is satisfied.

2. Let $A, B : X \to X'$ be nonlinear operators on a real Banach space X. Show that
   (a) A satisfies $(\mathrm{S})_+$ and B strongly continuous $\implies$ $A+B$ satisfies $(\mathrm{S})_+$,

Solution: Assume the left-hand side of $(\mathrm{S})_+$ holds for $A+B$; i.e.,

$$u_n \rightharpoonup u, \quad \limsup_{n\to\infty} \langle (A+B)u_n - (A+B)u, u_n - u\rangle \le 0.$$

We first note that as A is strongly continuous that

$$\limsup_{n\to\infty} |\langle Bu_n - Bu, u_n - u\rangle| \le \limsup_{n\to\infty} \|Bu_n - Bu\| \limsup_{n\to\infty} \|u_n - u\| = 0. \tag{2.1}$$

Hence,

$$\begin{aligned} 0 &\ge \limsup_{n\to\infty} \langle (A+B)u_n - (A+B)u, u_n - u\rangle \\ &= \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle + \limsup_{n\to\infty} \langle Bu_n - Bu, u_n - u\rangle \\ &= \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle, \end{aligned}$$

which means that the right-hand side of $(\mathrm{S})_+$ is satisfied for A. Then, by $(\mathrm{S})_+$ for A we have that $u_n \to u$, which is the right-hand side of $(\mathrm{S})_+$ for $A+B$.

   (b) A satisfies (S) and B strongly continuous $\implies$ $A+B$ satisfies (S),

Solution: Assume the left-hand side of (S) holds for $A+B$; i.e.,

$$u_n \rightharpoonup u, \quad \lim_{n\to\infty} \langle (A+B)u_n - (A+B)u, u_n - u\rangle = 0.$$

By identical result to (2.1)

$$0 = \lim_{n\to\infty} \langle (A+B)u_n - (A+B)u, u_n - u\rangle = \lim_{n\to\infty} \langle Au_n - Au, u_n - u\rangle;$$

hence, right-hand side of (S) is satisfied for A. Then, by (S) for A we have that $u_n \to u$, which is the right-hand side of (S) for $A+B$.

   (c) A satisfies (M) and B strongly continuous $\implies$ $A+B$ satisfies (M).

Solution: Assume the left-hand side of (M) holds for $A+B$; i.e.,

$$u_n \rightharpoonup u, \quad (A+B)u_n \rightharpoonup b, \quad \limsup_{n\to\infty} \langle (A+B)u_n, u_n\rangle \le \langle b, u\rangle.$$

We want to show the three assumptions of (M) are met for A. Clearly the first assumption $u_n \rightharpoonup u$ holds. Then, as B is strongly continuous,

$$\begin{aligned} \langle (A+B)u_n, v\rangle &\to \langle b, v\rangle && \forall v \in X \\ \langle Au_n, v\rangle + \langle Bu_n, v\rangle &\to \langle b, v\rangle && \forall v \in X \\ \langle Au_n, v\rangle + \langle Bu, v\rangle &\to \langle b, v\rangle && \forall v \in X \\ \langle Au_n, v\rangle &\to \langle b - Bu, v\rangle && \forall v \in X; \end{aligned}$$

therefore, $Au_n \rightharpoonup b - Bu \in X'$. This satisfies the second assumption. Finally,

$$
\begin{aligned}
\langle b, u\rangle &\geq \limsup_{n\to\infty}\langle (A+B)u_n, u_n\rangle \\
\lim_{n\to\infty}\langle (A+B)u_n, u\rangle &\geq \limsup_{n\to\infty}\langle Au_n, u_n\rangle + \limsup_{n\to\infty}\langle Bu_n, u_n\rangle \\
\lim_{n\to\infty}\langle Au_n, u\rangle &\geq \limsup_{n\to\infty}\langle Au_n, u_n\rangle + \limsup_{n\to\infty}\langle Bu_n, u_n - u\rangle \\
\langle b - Bu, u\rangle &\geq \limsup_{n\to\infty}\langle Au_n, u_n\rangle + \limsup_{n\to\infty}\langle Bu_n - Bu, u_n - u\rangle \\
&\quad + \lim_{n\to\infty}\langle Bu, u_n - u\rangle
\end{aligned}
$$

By strong continuity of B, cf. (2.1), and $u_n \rightharpoonup u$, we have that the third assumption of (M) is satisfied; i.e.,

$$\langle b - Bu, u\rangle \geq \limsup_{n\to\infty}\langle Au_n, u_n\rangle.$$

Hence, by (M) for A, $Au = b - Bu$ which gives that the right-hand side of (M) $(A+B)u = b$ is satisfied.

3. Let $A, B : X \to X'$ be nonlinear operators on a real Banach space X. Show that
   (a) A monotone and hemicontinuous $\implies$ A pseudomonotone,

Solution: By monotonicity

$$\langle Au_n - Au, u_n - u\rangle \geq 0 \qquad \implies \qquad \langle Au_n, u_n - u\rangle \geq \langle Au, u_n - u\rangle \to 0;$$

therefore,

$$0 \geq \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \geq \liminf_{n\to\infty}\langle Au_n, u_n - u\rangle \geq \lim_{n\to\infty}\langle Au, u_n - u\rangle = 0,$$

which gives that

$$\lim_{n\to\infty}\langle Au_n, u_n - u\rangle = 0.$$

Let $z = u + t(w-u)$, $t > 0$; then, by monotonicity,

$$
\begin{aligned}
&\langle Au_n - Az, u_n - z\rangle \geq 0 \\
&\qquad t\langle Au_n, u - w\rangle \geq \langle -Au_n, u_n - u\rangle + \langle Az, u_n - u\rangle + t\langle Az, u - w\rangle.
\end{aligned}
$$

As, by left-hand side of pseudomonotone $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0$ we have that

$$\langle Az, u - w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \qquad \forall w \in X.$$

As A is hemicontinuous, let $t \to 0$,

$$\langle Au, u - w\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \qquad \forall w \in X.$$

Hence, the right-hand side of pseudomonotone is shown.

   (b) A strongly continuous $\implies$ A pseudomonotone,

Solution: As A strong continuous then $u_n \rightharpoonup u \implies Au_n \to Au$ and by similar result to (2.1)

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle Au_n - Au, u_n - u\rangle \\
0 &= \lim_{n\to\infty} \langle Au_n, u_n\rangle - \lim_{n\to\infty} \langle Au_n, u\rangle - \lim_{n\to\infty} \langle Au, u_n - u\rangle \\
0 &= \lim_{n\to\infty} \langle Au_n, u_n\rangle - \langle Au, u\rangle \\
\langle Au, u\rangle &= \lim_{n\to\infty} \langle Au_n, u_n\rangle \\
\langle Au, u\rangle - \langle Au, w\rangle &= \lim_{n\to\infty} \langle Au_n, u_n\rangle - \lim_{n\to\infty} \langle Au_n, w\rangle && \forall w \in X \\
\langle Au, u - w\rangle &= \lim_{n\to\infty} \langle Au_n, u_n - w\rangle && \forall w \in X.
\end{aligned}$$

This satisfies the right-hand side of the pseudomonotone condition.

(c) A demicontinuous and satisfies $(\mathrm{S})_+ \implies A$ pseudomonotone,

Solution: As $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0 \qquad \implies \qquad \limsup_{n\to\infty} \langle Au_n - Au, u_n - u\rangle \le 0.$$

This satisfies the left-hand side of $(\mathrm{S})_+$; hence, $u_n \to u$. As A is demicontinuous this implies that $Au_n \rightharpoonup Au$. Hence, identically to the previous question as similar result to (2.1) holds for either strong convergence of u_n or Au_n,

$$\langle Au, u - w\rangle = \lim_{n\to\infty} \langle Au_n, u_n - w\rangle \qquad \forall w \in X,$$

which satisfies the right-hand side of the pseudomonotone condition.

(d) A pseudomonotone and B pseudomonotone $\implies A + B$ pseudomonotone,

Solution: We have that $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty} \langle (A+B)u_n, u_n - u\rangle \le 0.$$

Assume that there exists a subsequence $\{u_n\}$ such that

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle = a > 0 \qquad \implies \qquad \limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \le -a < 0.$$

As B is pseudomonotone,

$$\langle Bu, u - w\rangle \le \liminf_{n\to\infty} \langle Bu_n, u_n - w\rangle, \qquad \forall w \in X;$$

hence, setting $w = u$,

$$0 = \langle Bu, 0\rangle = \langle Bu, u-u\rangle \le \liminf_{n\to\infty} \langle Bu_n, u_n - u\rangle \le \limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \le -a < 0.$$

This is a contradiction; hence,

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0;$$

similarly,
$$\limsup_{n\to\infty}\langle Bu_n, u_n - u\rangle \le 0.$$
As A and B are pseudomonotone,
$$\langle Au, u-w\rangle \le \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \qquad \forall w \in X,$$
$$\langle Bu, u-w\rangle \le \liminf_{n\to\infty}\langle Bu_n, u_n - w\rangle \qquad \forall w \in X.$$
Then, $A+B$ is pseudomonotone as
$$\langle (A+B)u, u-w\rangle \le \liminf_{n\to\infty}\langle (A+B)u_n, u_n - w\rangle \qquad \forall w \in X.$$

(e) A pseudomonotone $\implies$ A satisfies (P),

Solution: Let $u_n \rightharpoonup u$ and assume that
$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle < 0;$$
i.e., A does not satisfy (P). Then, as A is pseudomontone,
$$\langle Au, u-w\rangle \le \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \le \limsup_{n\to\infty}\langle Au_n, u_n - w\rangle \qquad \forall w \in X.$$
Let $w = u$; then,
$$0 = \langle Au, u-u\rangle \le \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle,$$
which is a contradiction.

(f) A pseudomonotone $\implies$ A satisfies (M).

Solution: Assume the left-hand side of the condition (M) holds; i.e.,
$$u_n \rightharpoonup u, \quad Au_n \rightharpoonup b, \quad \limsup_{n\to\infty}\langle Au_n, u_n\rangle \le \langle b, u\rangle.$$
Then,
$$\limsup_{n\to\infty}\langle Au_n, u_n\rangle - \langle b, u\rangle \le 0$$
$$\limsup_{n\to\infty}\langle Au_n, u_n\rangle - \lim_{n\to\infty}\langle Au_n, u\rangle \le 0$$
$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0;$$
So, the left-hand side of the pseudomonotone condition is met; hence,
$$\langle Au, u-w\rangle \le \liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \qquad \forall w \in X$$
$$\langle Au, u-w\rangle \le \langle b, u\rangle - \langle b, w\rangle \qquad \forall w \in X.$$
Set $w = u - v$, and
$$\langle Au, v\rangle \le \langle b, v\rangle \qquad \forall v \in X;$$
hence, $Au = b$.