Nonlinear Differential Equations

Practical 5: Monotone Differential Equations

1. For $k \in \mathbb{N}$ and bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, let

$$(\mathcal{A}u)(\boldsymbol{x}) = \sum_{|\alpha| \le k} (-1)^\alpha \partial^\alpha a_\alpha(\boldsymbol{x}, \delta_k u(\boldsymbol{x}))$$

with coefficient functions $a_\alpha : \Omega \times \mathbb{R}^\kappa \to \mathbb{R}$ satisfying the Carathéodory (B1) and growth (B2) conditions from the lecture for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$. Then, we define the nonlinear form $a : W_0^{k,p}(\Omega) \times W_0^{k,p}(\Omega) \to \mathbb{R}$ and operator $A : W_0^{k,p}(\Omega) \to W^{-k,q}(\Omega)$ as

$$\langle Au, v \rangle \coloneqq a(u, v) \coloneqq \int_\Omega \sum_{|\alpha| \le k} a_\alpha(\boldsymbol{x}, \delta_k u(\boldsymbol{x})) \partial^\alpha v \, \mathrm{d}\boldsymbol{x}.$$

   (a) Show that the Nemyckii operators

$$\mathcal{N}_\alpha : W_0^{k,p}(\Omega) \to L^q(\Omega), \qquad (\mathcal{N}_\alpha u)(\boldsymbol{x}) \coloneqq a_\alpha(\boldsymbol{x}, \delta_k u(\boldsymbol{x})),$$

   are bounded and continuous. Additionally, show that there exists a function $g \in L^q(\Omega)$, which is non-negative almost everywhere, for $1/p + 1/q = 1$, and positive constant $C > 0$ such that

$$|a(u, v)| \le C \left(\|g\|_{0,q} + \|u\|_{k,p}^{p/q} \right) \|v\|_{k,p} \qquad \forall u, v \in W_0^{k,p}(\Omega).$$

   (Lemma 2.14)

   (b) Show that A is bounded and demicontinuous (Lemma 2.15).

   (c) Show that A is monotone if, for all $\xi, \eta \in \mathbb{R}^\kappa$ and almost all $\boldsymbol{x} \in \Omega$,

$$\sum_{|\alpha| \le k} \left(a_\alpha(\boldsymbol{x}, \xi) - a_\alpha(\boldsymbol{x}, \eta) \right) (\xi_\alpha - \eta_\alpha) \ge 0, \tag{1.1}$$

   (Lemma 2.16)

   (d) A is strictly monotone if equality only holds in (1.1) for

$$\sum_{|\beta| \le k} |\xi_\beta - \eta_\beta| = 0;$$

   (Lemma 2.16).

2. Consider the following boundary value problem problem:

$$\begin{aligned} -\nabla \cdot (\mu(\boldsymbol{x}, \nabla u) \nabla u) + b(\boldsymbol{x}, u) &= f(\boldsymbol{x}) && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where Ω has Lipschitz boundary, $u : \Omega \to \mathbb{R}$ is the unknown function, and $f \in L^2(\Omega)$.

   (a) Define, for this problem,

i. the coefficient functions $a_\alpha(\boldsymbol{x}, \xi)$, $\xi \in \mathbb{R}^\kappa$, for all multi-indices α, where $|\alpha| \leq 1$, and

ii. the weak formulation and definition of the weak solution of the boundary value problem.

(b) Derive conditions for μ and b such that Theorem 2.19 holds; i.e., state conditions such that

i. A is monotone,

ii. A is strictly monotone,

iii. A is coercive, and

iv. a_α satisfies the growth condition (B2).

3. Define the weak formulation, and show that there exists a weak solution, for the following boundary value problem in the bounded Lipschitz domain $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$: For $2 \leq p < \infty$, $f \in L^q(\Omega)$, $1/p + 1/q = 1$, and $t \geq 0$, $t \in \mathbb{R}$

$$
\begin{aligned}
-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) + tu &= f, && \text{in } \Omega, \\
u &= 0, && \text{on } \partial\Omega.
\end{aligned}
$$

Additionally, state if the weak solution is unique.