Nonlinear Differential Equations

Practical 4: Monotone & Continuous Operators

1. Let X be a Banach space, and $A : X \to X'$ be a nonlinear operator. Then prove the following:

   (a) A strongly monotone $\implies$ A uniformly monotone

   Solution: A is strongly monotone; hence, $\langle Au - Av, u - v\rangle \geq M\|u - v\|^2$, where $L > 0$ is a constant. Define a function $a(t) = Mt$; then,

   $$\langle Au - Av, u - v\rangle \geq a(\|u - v\|)\|u - v\|,$$

   $a(0) = 0$, $\lim_{t\to\infty} a(t) = \infty$, and a is strictly increasing. Therefore, A is uniformly monotone.

   (b) A uniformly monotone $\implies$ A strictly monotone

   Solution: As A uniformly monotone $\langle Au - Av, u - v\rangle \geq a(\|u - v\|)\|u - v\|$ where $a(0) = 0$ and a is strictly increasing. Then,

   $$u \neq v \quad \implies \quad \|u - v\| > 0 \quad \implies \quad a(\|u - v\|)\|u - v\| > 0.$$

   Therefore, $\langle Au - Av, u - v\rangle \geq a(\|u - v\|)\|u - v\| > 0$ for $u \neq v$ and, hence, A is strictly monotone.

   (c) A strictly monotone $\implies$ A monotone

   Solution: For $u \neq v$ $\langle Au - Av, u - v\rangle > 0$ as A is strictly monotone. If $u = v$; then $Au = Av$ and hence

   $$\langle Au - Av, u - v\rangle = \langle 0, 0\rangle = 0.$$

   Therefore, $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in X$.

   (d) A uniformly monotone $\implies$ A (nonlinear) coercive

   Solution:

   $$\begin{aligned}\lim_{\|u\|\to+\infty} \frac{\langle Au, u\rangle}{\|u\|} &= \lim_{\|u\|\to+\infty} \left(\frac{\langle Au - A0, u - 0\rangle - \langle A0, u\rangle}{\|u\|}\right) \\ &= \lim_{\|u\|\to+\infty} \frac{\langle Au - A0, u - 0\rangle}{\|u\|} - \lim_{\|u\|\to+\infty} \frac{\langle A0, u\rangle}{\|u\|}\end{aligned}$$

   By dual norm definition

   $$\|A0\| = \sup_{v\in X} \frac{\langle A0, v\rangle}{\|v\|} \geq \frac{\langle A0, v\rangle}{\|v\|} \quad \text{for all } v \in X \qquad \implies \qquad \|A0\|\|v\| \geq \langle A0, v\rangle;$$

and by fact that A is uniformly monotone

$$\begin{aligned}\lim_{\|u\|\to+\infty}\frac{\langle Au,u\rangle}{\|u\|} &\geq \lim_{\|u\|\to+\infty}\frac{a(\|u-0\|)\|u-0\|}{\|u\|} - \lim_{\|u\|\to+\infty}\frac{\|A0\|\|u\|}{\|u\|}\\ &= \lim_{\|u\|\to+\infty} a(\|u\|) - \|A0\|\\ &= +\infty.\end{aligned}$$

Hence, A is coercive.

(e) A uniformly monotone $\implies$ A stable

Solution: For $u \neq v$

$$\|Au-Av\| = \sup_{v\in X}\frac{\langle Au-Av,v\rangle}{\|v\|} \geq \frac{\langle Au-Av,u-v\rangle}{\|u-v\|} \geq \frac{a(\|u-v\|)\|u-v\|}{\|u-v\|}.$$

as A is uniformly monotone. Hence, $\|Au-Av\| \geq a(\|u-v\|)$ where a is strictly increasing, $a(0)=0$ and $\lim_{t\to\infty} a(t) = +\infty$.
For $u = v$ then $\|Au-Av\| = \|0\| = 0 = a(0) = a(\|0\|) = a(\|u-v\|)$.

(f) A Lipschitz continuous $\implies$ A continuous

Solution: Given a sequence $\{u_n\}$ such that $u_n \to u$ then $\lim_{n\to\infty}\|u_n-u\| = 0$. Then,

$$\lim_{n\to\infty}\|Au_n - Au\| \leq \lim_{n\to\infty} L\|u_n-u\| = 0;$$

hence, the sequence $Au_n \to Au$.

(g) A strongly continuous $\implies$ A continuous

Solution: Given a sequence $\{u_n\}$ we note that $u_n \to u \implies u_n \rightharpoonup u$ as for all $\ell \in X'$

$$\langle \ell, u_n - u\rangle \leq \|\ell\|\|u_n-u\| \to 0.$$

Then, from strong continuity $u_n \rightharpoonup u \implies Au_n \to Au$. Combining theses gives that $u_n \to u \implies Au_n \to Au$; hence, A is continuous.

(h) A strongly continuous $\implies$ A weakly continuous

Solution: Given a sequence $\{u_n\}$, strong continuity and fact weak convergence implies strong convergence gives that

$$u_n \rightharpoonup u \implies Au_n \to Au \implies Au_n \rightharpoonup Au;$$

hence, A is weakly continuous.

(i) A weakly continuous $\implies$ A demicontinuous

Solution: Given a sequence $\{u_n\}$, weakly continuity and fact weak convergence implies strong convergence gives that

$$u_n \to u \implies u_n \rightharpoonup u \implies Au_n \rightharpoonup Au;$$

hence, A is demicontinuous.

(j) A continuous $\implies$ A demicontinuous

Solution: Given a sequence $\{u_n\}$, continuity and fact weak convergence implies strong convergence gives that

$$u_n \to u \implies Au_n \to Au \implies Au_n \rightharpoonup Au;$$

hence, A is demicontinuous.

(k) A demicontinuous $\implies$ A hemicontinuous

Solution: If A is demicontinuous; then, for a sequence $\{z_n\}$

$$z_n \to z \quad \implies \quad Az_n \rightharpoonup Az \quad \iff \quad \langle Az_n - Az, v\rangle \to 0 \quad \text{for all } v \in X.$$

Select $s_n, s \in [0,1]$ such that $s_n \to s$ and set $z_n = u + s_n v$; then,

$$\lim_{n\to\infty} \langle A(u + s_n v), w\rangle = \langle A(u + sv), w\rangle \qquad \text{for all } w \in X.$$

Hence, hemicontinuity follows as $\langle A(u + tv), w\rangle$ is continuous for $t \in [0,1]$ if and only if

$$\lim_{n\to\infty} \langle A(u + s_n v), w\rangle = \langle A(u + sv), w\rangle \qquad \text{for all } w \in X.$$

2. Let X be a Banach space, and $A, B : X \to X'$ be nonlinear operators. Then prove the following:

(a) A strongly monotone and B strongly monotone $\implies$ $A + B$ strongly monotone

Solution: As A and B are strongly monotone; then, there exists positive constants M_A and M_B such that

$$\langle Au - Av, u - v\rangle \geq M_A \|u - v\|,$$
$$\langle Bu - Bv, u - v\rangle \geq M_B \|u - v\|.$$

Then,

$$\begin{aligned}\langle (A+B)u - (A+B)v, u - v\rangle &= \langle Au - Av, u - v\rangle + \langle Bu - Bv, u - v\rangle \\ &\geq (M_A + M_B)\|u - v\|^2.\end{aligned}$$

Then, $A + B$ is strongly monotone with constant $M_A + M_B$.

(b) A strongly monotone and B monotone $\implies$ $A + B$ strongly monotone

Solution: As A is strongly monotone and B monotone; then, there exists a positive constant M_A such that

$$\begin{aligned}
&\langle Au - Av, u - v\rangle \geq M_A \|u - v\|. \\
&\langle Bu - Bv, u - v\rangle \geq 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\langle (A + B)u - (A + B)v, u - v\rangle &= \langle Au - Av, u - v\rangle + \underbrace{\langle Bu - Bv, u - v\rangle}_{\geq 0} \\
&\geq \langle Au - Av, u - v\rangle \\
&\geq M_A \|u - v\|^2.
\end{aligned}$$

Then, $A + B$ is strongly monotone with constant M_A.

3. Let $\boldsymbol{A} : \mathbb{R}^m \to \mathbb{R}^m$, $m > 0$, be a symmetric positive definite matrix. Show that the operator $A : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$\langle Au, v\rangle = (\boldsymbol{A}u) \cdot v \qquad \text{for all } v \in \mathbb{R}^m$$

is strongly monotone and Lipschitz continuous.

Hint. *Consider the eigendecomposition of* $\boldsymbol{A}$*.*

Solution: As $\boldsymbol{A}$ is symmetric then we can decompose it as $\boldsymbol{A} = Q\Lambda Q^\top$, where Q is orthogonal and Λ is diagonal with $\Lambda_{ii} = \lambda_i$, $i = 1, \ldots, m$ with λ_i is the ith eigenvalue of A. Additionally, as $\boldsymbol{A}$ is symmetric positive definite, for the eigenvalue λ_i and matching eigenvector $x_i \in \mathbb{R}^m$

$$0 < x_i^\top (\boldsymbol{A} x_i) = x_i^\top (\lambda_i x_i) = \lambda_i x_i^T x_i = \lambda \|x_i\|^2.$$

As $\|x_i\| \geq 0$ then $\lambda_i > 0$ for all $\lambda_i \in \sigma(\boldsymbol{A})$. So all eigenvalues are strictly positive; hence, $\min_{\lambda \in \sigma(\boldsymbol{A})} \lambda_i > 0$. We can now show that A is strongly monotone and Lipschitz continuous using the natural vector 2-norm $\|\cdot\|_2$ on $\mathbb{R}^m$:

strongly monotone:

$$\begin{aligned}
\langle Au - Av, u - v\rangle &= (\boldsymbol{A}u - \boldsymbol{A}v)^\top (u - v) \\
&= (u - v)^\top \boldsymbol{A}^\top (u - v) \\
&= (u - v)^\top Q^\top \Lambda Q (u - v) \\
&= \sum_{i=1}^m \left((u - v)^\top Q^\top\right)_i \lambda_i \left(Q(u - v)\right)_i \\
&\geq \left(\min_{\lambda \in \sigma(\boldsymbol{A})} \lambda\right) \sum_{i=1}^m \left((u - v)^\top Q^\top\right)_i \left(Q(u - v)\right)_i \\
&= \left(\min_{\lambda \in \sigma(\boldsymbol{A})} \lambda\right) (u - v)^\top Q^\top Q (u - v) \\
&= M\|u - v\|_2^2
\end{aligned}$$

where $Q^\top Q = I$ as Q is orthogonal, and $M = \min_{\lambda \in \sigma(\boldsymbol{A})} \lambda$.

Lipschitz continuous:

$$
\begin{aligned}
\|\boldsymbol{A}u-\boldsymbol{A}v\|_2^2 &= (\boldsymbol{A}u-\boldsymbol{A}v)^\top(\boldsymbol{A}u-\boldsymbol{A}v) \\
&= (u-v)^\top Q^T\Lambda Q(\boldsymbol{A}u-\boldsymbol{A}v) \\
&\le \left(\max_{\lambda\in\sigma(\boldsymbol{A})}\lambda\right)(u-v)^\top Q^T Q(\boldsymbol{A}u-\boldsymbol{A}v) \\
&= \left(\max_{\lambda\in\sigma(\boldsymbol{A})}\lambda\right)(u-v)^\top(\boldsymbol{A}u-\boldsymbol{A}v) \\
&\le L\|u-v\|_2\|\boldsymbol{A}u-\boldsymbol{A}v\|_2
\end{aligned}
$$

where $L=\max_{\lambda\in\sigma(\boldsymbol{A})}\lambda>0$. Divide both sides by $\|\boldsymbol{A}u-\boldsymbol{A}v\|$ completes the proof.

4. Let X be a Hilbert space, $A:X\to X'$ be strongly monotone and Lipschitz continuous, $f\in X'$ and J_X be the Riesz-isomorphism on X.
   (a) Show that there exists a constant ε such that the mapping $T:X\to X$ defined as
   $$T(u)=u-\varepsilon J_X^{-1}(Au-f)$$
   is strongly contractive; i.e,
   $$\|T(x)-T(y)\|\le k\|x-y\| \qquad \text{for all } x,y\in X$$
   with $k^2=1+\varepsilon^2L^2-2\varepsilon M$. Additionally, specify the condition on ε such that $k\in(0,1)$.

Solution: For simplicity we assume the inner product is symmetric. As A is strongly monotone and Lipschitz continuous, and by the definition of the Riesz-isomorphism,

$$
\begin{aligned}
\|T(u)-T(v)\|^2 &= (u-v-\varepsilon J_X^{-1}(Au-Av),u-v-\varepsilon J_X^{-1}(Au-Av)) \\
&= (u-v,u-v)-2\varepsilon(u-v,J_X^{-1}(Au-Av)) \\
&\quad +\varepsilon^2(J_X^{-1}(Au-Av),J_X^{-1}(Au-Av)) \\
&= \|u-v\|^2-2\varepsilon\langle Au-Av,u-v\rangle+\varepsilon^2\|Au-Av\|^2 \\
&\le \|u-v\|^2-2\varepsilon M\|u-v\|^2+\varepsilon^2L^2\|u-v\|^2 \\
&= k^2\|u-v\|^2.
\end{aligned}
$$

Taking the square root of both sides completes the proof.
We require that $k^2\ge 0$; hence, we require that

$$1+\varepsilon^2L^2-2\varepsilon M>0.$$

Note that as

$$M\|u-v\|^2\le\langle Au-Av,u-v\rangle\le\|Au-Av\|\|u-v\|\le L\|u-v\|^2$$

we have that $M<L$; hence,

$$1+\varepsilon^2L^2-2\varepsilon M\ge 1+\varepsilon^2L^2-2\varepsilon L=(1-\varepsilon L)^2>0$$

providing $\varepsilon L \neq 1$; i.e., $\varepsilon \neq L^{-1}$. Additionally, we require that $k^2 < 1$; hence, we require that

$$\varepsilon^2 L^2 - 2\varepsilon M < 0.$$

If we consider $\varepsilon^2 L^2 - 2\varepsilon M = 0$ we have that

$$\varepsilon(\varepsilon L^2 - 2M) = 0;$$

hence, $\varepsilon^2 L^2 - 2\varepsilon M = 0$ when $\varepsilon = 0$ or $\varepsilon = 2L^{-2}$; therefore, $\varepsilon^2 L^2 - 2\varepsilon M < 0$ if $\varepsilon \in (0, 2ML^{-2})$.

(b) Compute the optimal value of ε such that the iteration

$$u_{m+1} = u_m - \varepsilon J_X^{-1}(Au_m - f)$$

converges fastest to the *unique* solution of $Au = f$ and the compute the contraction constant k

Hint. *From Corollary 2.9 and Banach's fixed point theorem the error is given by*

$$\|u - u_m\| \leq \frac{k^m}{1-k}\|x_0 - x_1\|;$$

hence, the fastest convergence rate is given when k is close to zero.

Solution: We want to minimise the constant k; i.e, defining

$$\varphi(\varepsilon) := 1 + \varepsilon^2 L^2 - 2\varepsilon M$$

we want to minimise $\varphi(\varepsilon)$; then, we consider

$$0 = \varphi'(\varepsilon) = 2\varepsilon L^2 - 2M \qquad \Longrightarrow \qquad \varepsilon = \frac{M}{L^2}.$$

Therefore,

$$k^2 = \varphi(\varepsilon) = 1 - \frac{M^2}{L^2}.$$

Note, that this is only valid if $M \neq L$; otherwise $k = 0$.