Nonlinear Differential Equations

Practical 3: Banach & Sobolev Spaces

1. Prove the *generalised Hölder's inequality:* For $m \in \mathbb{N}$, $m \geq 2$, let there exists functions f_i, $i = 1, \ldots, m$ and $0 \leq p_1, \ldots, p_m \leq \infty$; then,

$$\|f_1 \cdots f_m\|_{0,r} \leq \|f_1\|_{0,p_1} \cdots \|f_m\|_{0,p_m} \qquad \text{for } \frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}.$$

Hint. *Use the standard Hölder's inequality (Lemma 1.12) and induction.*

2. Let $\Omega \subset \mathbb{R}^2$ be a measurable domain with Lipschitz boundary and $\alpha \in \mathbb{N}_0^n$ be a multi-index; then, prove that the seminorm

$$|v|_{1,2,\Omega} = \left(\sum_{|\alpha|=1} \|\partial^\alpha v\|_{0,2,\Omega}^2 \right)^{1/2}$$

is a norm on the space $H_0^1(\Omega)$.

3. Let $F : C^2([0, L]) \to C([0, L])$ be defined by

$$F(\varphi) = \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \lambda \sin \varphi$$

for fixed $\lambda \in \mathbb{R}$; cf. Example 1.1. Derive the Fréchet derivative in φ and Gâteaux derivative in φ in the direction ψ of F.

Hint. *Consider $F(\varphi + \psi) - F(\varphi)$ and $F(\varphi + t\psi) - F(\varphi)$, respectively, for $\varphi, \psi \in C^2([0, L])$ with small $\|\varphi\|_{2,\infty}$ and $\|\psi\|_{2,\infty}$.*

4. Let $\Omega \subset \mathbb{R}^n$ be a measurable domain with Lipschitz boundary and $X = H_0^1(\Omega)$; then, define $F : X \to X'$ be defined such that for $u, v \in X$

$$\langle F(u), v \rangle = \int_\Omega \mu(|\nabla u|) \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x},$$

where $\mu(t) \in C([0, \infty))$ is the *Carreau law* defined by

$$\mu(t) := \mu_{\text{inf}} + (\mu_0 - \mu_{\text{inf}}) \left(1 + (\lambda t)^2\right)^{\frac{n-1}{2}}$$

for constants $\mu_{\text{inf}}, \mu_0, n, \lambda \in \mathbb{R}$. Compute

$$\langle F_G'(u, w), v \rangle$$

where $F_G'(u, w)$ is the Gâteaux derivative of F in u in the direction w.

Hint. *Use*

$$\langle F_G'(u, w), v \rangle = \lim_{t \to 0} \frac{\langle F(u + tw) - F(u), v \rangle}{t}.$$