Nonlinear Differential Equations

Practical 2: Compact Operators

1. Let X, Y, and Z be Banach spaces, $L_1 \in \mathcal{L}(X, Y)$, and $L_2 \in \mathcal{L}(Y, Z)$; then, show that if L_1 is compact then $L_2 \circ L_1$ is also compact. Additionally, show that if L_2 is compact then $L_2 \circ L_1$ is compact.

Solution:

- If L_1 is compact then every bounded sequence $\{x_n\} \subset \mathcal{D}(L_1) \subset X$ has a convergent subsequence in Y; i.e., there exists a sequence $\{L_1(x_{n_k})\}$ such that

  $$\lim L_1(x_{n_k}) = y \in Y.$$

  Additionally, as L_2 is a continuous linear map the limit of the sequence applying L_2 to this sequence has limit $L_2(y)$; i.e.,

  $$\lim L_2(L_1(x_{n_k})) = L_2(y) \in Z.$$

  Therefore, every bounded sequence in $\{x_n\} \subset \mathcal{D}(L_2 \circ L_1)$ has a convergent subsequence $\{L_2(L_1(x_{n_k}))\}$ in Z; hence, $L_2 \circ L_1$ is a compact operator.

- As L_1 is a continuous linear operator it is a bounded operator; then, the bounded subset $K \subset \mathcal{D}(L_1)$ is mapped to a bounded subset $L_1(K) \subset Y$. Then as L_2 is compact this bounded susbset is mapped to a precompact subset of Z. Therefore, the image of a bounded subset $K \subset \mathcal{D}(L_2 \circ L_1)$ under $L_2 \circ L_1$ is a precompact subset of Z; hence, $L_2 \circ L_1$ is a compact operator.

2. Let X and Y be Banach spaces. Show that a compact linear operator $L \in \mathcal{L}(X, Y)$ maps every weakly convergent sequence into a strongly convergent sequence.

Solution: Let there exists a sequence $x_n \rightharpoonup x$; then, $\ell(x_n) \to \ell(x)$ for all $\ell \in X'$. Define $\widehat{x}_n \in X''$ by $\widehat{x}_n(\ell) = \ell(x_n)$ for all $\ell \in X'$. For a fixed $\ell \in X'$ the sequence $\{\ell(x_n)\}$ is a *bounded* strongly convergent sequence; i.e., $\sup\|\ell(x_n)\| < \infty \implies \sup\|\widehat{x}_n(\ell)\| < \infty$. By *uniform boundedness principal*

$$\sup\|\widehat{x}_n(\ell)\| < \infty \; \forall \ell \in X' \quad \iff \quad \sup\|\widehat{x}_n\| < \infty.$$

Hence, $\sup\|x_n\| = \sup\|\widehat{x}_n\| < \infty$ which implies the sequence x_n is bounded.

Define $y_n = Lx_n$ and $y = Lx$; then

$$\langle \ell, y_n \rangle - \langle \ell, y \rangle = \langle \ell, L(x_n - x) \rangle = \langle L^d \ell, x_n - x \rangle$$

for any $\ell \in Y'$. As $L^d \ell \in X'$ then $\langle L^d \ell, x_n - x \rangle \to 0$; hence $y_n \rightharpoonup y$.

Finally, we assume that $y_n \not\to y$ strongly. Then, $\exists \varepsilon > 0$ and a subsequence $\{y_{n_k}\}$ such that $\|y_{n_k} - y\| \geq \varepsilon$. The sequence $\{x_{n_k}\}$ is bounded and L is compact; hence, a convergence subsequence of $\{y_{n_k}\}$ exists in Y with limit $\widetilde{y} \neq y$. Strong convergence implies weak convergence, so this subsequence weakly converges to $\widetilde{y}$, but this contradicts the fact that $y_n \rightharpoonup y \neq \widetilde{y}$. Hence, by contradiction, $y_n \to y$.

3. For the Banach spaces X, Y, and $L \in \mathcal{L}(X, Y)$ then the *dual operator* $L^d \in \mathcal{L}(Y', X')$ is uniquely determined for every $\ell \in Y'$, $\ell \in Y' \mapsto L^d\ell \in X'$, by

$$\langle \ell, Lx \rangle_{Y' \times Y} = \langle L^d \ell, x \rangle_{X' \times X} \qquad \text{for all } x \in X.$$

Show that a linear operator $L \in \mathcal{L}(X, Y)$ is compact if and only if the dual operator $L^d \in \mathcal{L}(Y', X')$ is compact.

Solution:

L **compact** $\implies$ L^d **compact** Let $\{\ell\} \subset Y'$, K be a bounded subset of X, and $\widehat{K} = \overline{L(K)}$. Then $f_n = \ell|_{\widehat{K}} \in C(\widehat{K})$ is bounded and equicontinuous, so by *Arzelà-Ascoli* it has a convergent subsequence $\{f_{n_k}\}$ in $C(\widehat{K})$. Then

$$\|L^d \ell_{n_i} - L^d \ell_{n_k}\| = \sup_{x \in K} \|\ell_{n_i}(x) - \ell_{n_k}(x)\| = \sup_{X \in \widehat{K}} \|f_{n_i}(x) - f_{n_k}(x)\|.$$

Hence, $\{L^d \ell\}$ is a Cauchy sequence and, hence, converges $\implies$ L^d is compact.