Nonlinear Differential Equations

Practical 2: Compact Operators

1. Let X, Y, and Z be Banach spaces, $L_1 \in \mathcal{L}(X, Y)$, and $L_2 \in \mathcal{L}(Y, Z)$; then, show that if L_1 is compact then $L_2 \circ L_1$ is also compact. Additionally, show that if L_2 is compact then $L_2 \circ L_1$ is compact.

2. Let X and Y be Banach spaces. Show that a compact linear operator $L \in \mathcal{L}(X, Y)$ maps every weakly convergent sequence into a strongly convergent sequence.

3. For the Banach spaces X, Y, and $L \in \mathcal{L}(X, Y)$ then the *dual operator* $L^d \in \mathcal{L}(Y', X')$ is uniquely determined for every $\ell \in Y'$, $\ell \in Y' \mapsto L^d \ell \in X'$, by

$$\langle \ell, Lx \rangle_{Y' \times Y} = \langle L^d \ell, x \rangle_{X' \times X} \qquad \text{for all } x \in X.$$

   Show that a linear operator $L \in \mathcal{L}(X, Y)$ is compact if and only if the dual operator $L^d \in \mathcal{L}(Y', X')$ is compact.