# Real Functions I

Daniel Campbell

# Contents


1. Preliminary Notation 4
2. Vitali Covering Theorems 4
3. The maximal operator 7
4. The Besicovitch covering theorem 9
5. Differentiation of measures 12

## 1. Preliminary Notation

NOTATION 1.1.

- The symbol $\mathcal{B}$ denotes the family of all closed balls in $\mathbb{R}^n$. Balls must have positive diameter; singleton balls are excluded.
- As a rule, $B$ denotes an open ball and $\overline{B}$ a closed ball. This will be a little annoying, since we will keep working with closed balls here, but I want the notation to be consistent across my notes, so bear with me.
- If $B \subset \mathbb{R}^n$ is an open or closed ball and $\alpha > 0$, then $\alpha B$ denotes the open or closed ball, respectively, with the same centre as $B$ and with radius $\alpha$ times that of $B$.
- $\mathcal{L}^n$ denotes Lebesgue measure on $\mathbb{R}^n$.
- $\mathcal{L}^{n*}$ denotes Lebesgue outer measure on $\mathbb{R}^n$.
- $|\cdot|$ denotes the Euclidean norm.

DEFINITION 1.2. An *outer measure* on a set $X$ is a set function $\gamma : 2^X \to [0, \infty]$ such that

(1) $\gamma(\emptyset) = 0$;
(2) $A \subset E \implies \gamma(A) \leq \gamma(E)$;
(3) $\gamma\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \gamma(A_j)$.

DEFINITION 1.3. Let $\mathcal{R}_n$ be the family of all bounded $n$-dimensional intervals in $\mathbb{R}^n$. Consider a set function $\nu : \mathcal{R}_n \to [0, \infty)$ and a measure $(\mathcal{S}, \mu)$ obtained from $\nu$ by Carathéodory's construction, with $\mu = \nu$ on $\mathcal{R}_n$. We call $\mu$ a *Lebesgue–Stieltjes measure*, or an *LS measure* for short, generated by the set function $\nu$.

DEFINITION 1.4. Let $\mu$ and $\nu$ be two measures on $X$. We say that $\nu$ and $\mu$ are *mutually singular* if there exists $N \subseteq X$ such that $\nu(X \setminus N) = 0$ and $\mu(N) = 0$.

## 2. Vitali Covering Theorems

Covering theorems provide a tool that allows us, in the context of measure theory, to deduce global properties from local ones.

MINOR PROPOSITION 2.1. *Let $E \subset \mathbb{R}^n$ be measurable, and let $\mathcal{S}$ be a finite family of closed balls covering $E$. Then there exist $M \in \mathbb{N}$ (depending on $E$ and $\mathcal{S}$) and a family $\{\overline{B}_1, \overline{B}_2, \ldots, \overline{B}_M\}$ of pairwise disjoint balls such that*

$$E \subset \bigcup_{k=1}^{M} 3\overline{B}_k.$$

PROOF. Without loss of generality, we may assume that $\mathcal{S}$ is nonempty. Choose $\overline{B}_1 \in \mathcal{S}$ to have the largest radius among all balls in $\mathcal{S}$. Suppose that $\overline{B}_1, \ldots, \overline{B}_{k-1}$ have already been chosen. If such a ball exists, choose $\overline{B}_k \in \mathcal{S}$ disjoint from $\bigcup_{i<k} \overline{B}_i$ and with the largest radius among the balls in $\mathcal{S}$ having this property. This produces a finite sequence of closed balls $\overline{B}_1, \ldots, \overline{B}_M$. We stop when there are no balls left, or when none of the remaining balls has the required property.

We now prove that $E \subset \bigcup_{k=1}^{M} 3\overline{B}_k$. This is trivial if all balls have been selected, so assume otherwise. Let $x \in E$. There exists $\overline{B} \in \mathcal{S}$ such that $x \in \overline{B}$. Let $k$ be the smallest index such that $\overline{B} \cap \overline{B}_k \neq \emptyset$. Then $\operatorname{diam}(\overline{B}) \leq \operatorname{diam}(\overline{B}_k)$; otherwise, we would have selected $\overline{B}$ instead of $\overline{B}_k$. Consequently, $2\overline{B}_k$ must contain the centre of $\overline{B}$, and $\overline{B} \subset 3\overline{B}_k$. Thus $x \in 3\overline{B}_k$. □

LEMMA 2.2. *Let $X$ be a separable metric space, and let $\mathcal{V}$ be a family of closed balls. Then $\mathcal{V}$ contains a maximal disjoint subfamily $\mathcal{V}'$, and this subfamily is at most countable.*

PROOF. Consider a countable dense set $\{x_k\}$. Start with the empty family $\mathcal{V}_0$. At step $k$, try to add to $\mathcal{V}_{k-1}$ a ball that is disjoint from all balls in $\mathcal{V}_{k-1}$ and contains $x_k$. If such a ball $\overline{B}_k$ exists, set $\mathcal{V}_k = \mathcal{V}_{k-1} \cup \{\overline{B}_k\}$; otherwise, set $\mathcal{V}_k = \mathcal{V}_{k-1}$.

At the end, we obtain a countable disjoint family $\mathcal{V}' = \bigcup_k \mathcal{V}_k$. Suppose that another ball $\overline{B}$ could be added. This ball contains some $x_k$. Either we already have a ball containing $x_k$, which contradicts the possibility of adding a disjoint ball, or $\overline{B}$ has a nonempty intersection with one of the balls $\overline{B}_i$, $i = 1, \dots, k-1$, which again contradicts the possibility of adding it. Hence the family is maximal. □

THEOREM 2.3 (Vitali type). *Let $X$ be a separable metric space, and let $E \subset X$. Let $\mathcal{V}$ be a family of closed balls in $X$ covering $E$. Assume that*

$$R := \sup\{r : \overline{B}(x, r) \in \mathcal{V}\} < \infty.$$

*Then there exists a countable disjoint subfamily $\mathcal{V}'$ of $\mathcal{V}$ such that*

$$E \subset \bigcup_{\overline{B} \in \mathcal{V}'} 5\overline{B}.$$

PROOF. We may assume that $R = 1$; otherwise, we would simply insert the factor $R$ throughout the proof. Set

$$\mathcal{V}_m = \{\overline{B} \in \mathcal{V} : 2^{-m-1} < r \le 2^{-m}\}.$$

Start with the empty family. At step $m$, for $m \ge 0$, choose, using Lemma 2.2, a maximal disjoint subfamily $\mathcal{V}'_m$ of

$$\{\overline{B} \in \mathcal{V}_m : \overline{B} \text{ intersects no ball in } \mathcal{V}'_0 \cup \cdots \cup \mathcal{V}'_{m-1}\}.$$

Set

$$\mathcal{V}' = \bigcup_{m=0}^{\infty} \mathcal{V}'_m.$$

Let us prove that this family has the required covering property. Let $z \in E$. There exist $k$ and $\overline{B} \in \mathcal{V}_k$ such that $z \in \overline{B}(x, r)$. If $\overline{B}(x, r)$ was not selected for $\mathcal{V}'_k$, this must be because there exist $m \le k$ and $\overline{B}(y, \rho) \in \mathcal{V}'_m$ such that $\overline{B}(x, r) \cap \overline{B}(y, \rho) \ne \emptyset$. Thus

$$d(z, y) \le d(z, x) + d(x, y) \le r + (\rho + r) < 5\rho.$$

Indeed,

$$r \le 2^{-k} \le 2 \cdot 2^{-m-1} < 2\rho.$$

It follows that $z \in \overline{B}(y, 5\rho)$, so the covering property holds. □

DEFINITION 2.4. Let $A \subset \mathbb{R}^n$. We say that a family $\mathcal{V}$ of closed balls in $\mathbb{R}^n$ is a *Vitali cover* of $A$ if

$$\forall x \in A \ \forall \varepsilon > 0 \ \exists \overline{B} \in \mathcal{V} : x \in \overline{B} \quad \text{and} \quad \operatorname{diam} \overline{B} < \varepsilon.$$

THEOREM 2.5 (Vitali). *Let $A \subset \mathbb{R}^n$, and let $\mathcal{V}$ be a family of closed balls forming a Vitali cover of $A$. Then there exist pairwise disjoint balls $\{\overline{B}_1, \overline{B}_2, \dots\}$ selected from $\mathcal{V}$ such that*

$$\mathcal{L}^n\Big(A \setminus \bigcup_j \overline{B}_j\Big) = 0.$$

PROOF. We have

$$A = N \cup \bigcup_{k=0}^{\infty} \Big((A \cap B(0, k+1)) \setminus \overline{B}(0, k)\Big), \qquad \mathcal{L}^n(N) = 0.$$

If the assertion holds for each set $(A \cap B(0, k+1)) \setminus \overline{B}(0, k)$, then it holds for all of $A$. We may therefore assume that $A$ is contained in a bounded open set $G$. Select a countable

disjoint subfamily $\{\overline{B}_1, \overline{B}_2, \dots\}$ of balls from $\mathcal{V}$ that are also contained in $G$, in the same way as in the proof of Theorem 2.3. To be precise; set

$$\mathcal{V}_m = \{\overline{B} \in \mathcal{V} : 2^{-m-1} < r \leq 2^{-m}\}.$$

Start with the empty family. At step $m$, for $m \geq 0$, choose, using Lemma 2.2, a maximal disjoint subfamily $\mathcal{V}'_m$ of

$$\{\overline{B} \in \mathcal{V}_m : \overline{B} \text{ intersects no ball in } \mathcal{V}'_0 \cup \dots \cup \mathcal{V}'_{m-1}\}.$$

Set

$$\mathcal{V}' = \bigcup_{m=0}^{\infty} \mathcal{V}'_m.$$

From Theorem 2.3, if follows that $\{5\overline{B}_1, 5\overline{B}_2 \dots\}$ cover $A$ and therefore $\{5\overline{B}_{i+1}, 5\overline{B}_{i+2} \dots\}$ cover $A \setminus \bigcup_{j=1}^{i} \overline{B}_j$.

We have

$$\sum_{i=1}^{\infty} \mathcal{L}^n(\overline{B}_i), = \mathcal{L}^n\Big(\bigcup_{i=1}^{\infty} \overline{B}_i\Big) \leq \mathcal{L}^n(G) < \infty.$$

Hence the series $\sum_{i=1}^{\infty} \mathcal{L}^n(\overline{B}_i)$ converges, and consequently $\lim_i \operatorname{diam} \overline{B}_i = 0$.

Let $x$ be an uncovered point, that is, $x \in A \setminus \bigcup_j \overline{B}_j$. Then, as stated above it holds that

$$\forall i \in \mathbb{N}\ \exists j \in \mathbb{N},\ j > i : x \in 5\overline{B}_j.$$

Thus

$$\begin{aligned}\mathcal{L}^{n*}\Big(A \setminus \bigcup_j \overline{B}_j\Big) &\leq \mathcal{L}^n\Big(\bigcup_{j=i}^{\infty} 5\overline{B}_j\Big) \\ &\leq \sum_{j=i}^{\infty} \mathcal{L}^n(5\overline{B}_j) = 5^n \sum_{j=i}^{\infty} \mathcal{L}^n(\overline{B}_j).\end{aligned}$$

Since $\lim_{i\to\infty} \sum_{j=i}^{\infty} \mathcal{L}^n(\overline{B}_j) = 0$, we obtain $\mathcal{L}^{n*}(A \setminus \bigcup_j \overline{B}_j) = 0$, and hence also $\mathcal{L}^n(A \setminus \bigcup_j \overline{B}_j) = 0$. □

Definition 2.6. We say that a measure $\mu$ on $\mathbb{R}^n$ *satisfies the Vitali covering theorem* if, for every $M \subset \mathbb{R}^n$ and every Vitali cover $\mathcal{V}$ of $M$, there exists a countable disjoint cover $\mathcal{A} \subset \mathcal{V}$ such that $\mu(M \setminus \bigcup \mathcal{A}) = 0$.

Remark 2.7.

(1) By Theorem 2.5, $\mathcal{L}^n$ satisfies the Vitali covering theorem. (A direct application.)
(2) If $\mu$ satisfies the Vitali covering theorem and $\nu \ll \mu$, then $\nu$ satisfies the Vitali covering theorem. (A direct consequence of absolute continuity.)
(3) If $\mu$ is a Borel measure on $\mathbb{R}^2$ such that $\mu(A) = \mathcal{L}^1(A \cap (\mathbb{R} \times \{0\}))$ for every Borel set $A \subset \mathbb{R}^2$, then the Vitali covering theorem does not hold for $\mu$: the line can be covered by balls each of which touches it at just one point. Nevertheless, since this measure is Radon, it satisfies the centred Vitali covering theorem. We will prove a centred version of the Vitali theorem later.

Question 2.8. Prove that: for every $G \subset \mathbb{R}^n$ open and for every $\epsilon > 0$, there is a closed (in $\mathbb{R}^n$) set $K \Subset G$ such that

$$\mathcal{L}^n(G) \leq (1 - \varepsilon)\mathcal{L}^n(K).$$

QUESTION 2.9. Prove that: there exists a $C > 0$ such that for every $G \subset \mathbb{R}^n$ with finite measure there exists a finite collection of disjoint closed balls $\{\overline{B}_1, \overline{B}_2, \dots, \overline{B}_M\}$ satisfying $\bigcup_{i=1}^M \overline{B}_i \subset G$ and

$$\mathcal{L}^n(G) \leq C \sum_{i=1}^M \mathcal{L}^n(\overline{B}_i).$$

QUESTION 2.10. Is there a counter example to Theorem 2.3 if the bounded diameter condition is not satisfied?

QUESTION 2.11. Can we replace closed balls with open balls in Theorem 2.3 and Theorem 2.5?

QUESTION 2.12. Can we replace closed balls with closed convex sets with non-empty interior in Theorem 2.3 and Theorem 2.5?

QUESTION 2.13. Given that the constant in Theorem 2.3 is equal to 3, can the constant 5 from Theorem 2.5 be improved?

QUESTION 2.14. Is it possible to bound the overlap of the covering given by Theorem 2.3? I.e. is there an $M \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} \chi_{5\overline{B}_i} \leq M$$

on $X$?

## 3. The maximal operator

DEFINITION 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable. For $x \in \mathbb{R}^n$, define

$$Mf(x) = \sup_{\overline{B} \ni x} \frac{1}{\mathcal{L}^n(\overline{B})} \int_{\overline{B}} |f| \, d\mathcal{L}^n$$

and

$$M_c f(x) = \sup_{r>0} \frac{1}{\mathcal{L}^n(\overline{B(x,r)})} \int_{B(x,r)} |f| \, d\mathcal{L}^n.$$

REMARK 3.2. In the definition we can replace $\overline{B}$ with $B$ and nothing changes.

LEMMA 3.3. *Let $f : \mathbb{R}^n \to [0, \infty)$ be $\mathcal{L}^n$-measurable. Then*

$$\int_{\mathbb{R}^n} f^p \, d\mathcal{L}^n = \int_0^\infty p t^{p-1} \mathcal{L}^n\Big(\{x \colon f(x) > t\}\Big) \, dt.$$

PROOF. Set

$$W = \{(x, s) \in \mathbb{R}^{n+1} : 0 < s < f(x)\}.$$

Then

$$W = \bigcup_{q \in \mathbb{Q}^+} \{x \colon f(x) > q\} \times (0, q),$$

where $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$. Thus we see that $W$ is $\mathcal{L}^{n+1}$-measurable. By Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} f^p \, d\mathcal{L}^n &= \int_{\mathbb{R}^n} \left( \int_0^{f(x)} p t^{p-1} \, dt \right) d\mathcal{L}^n(x) \\ &= \int_0^\infty \left( \int_{\{x \colon f(x) > t\}} p t^{p-1} \, d\mathcal{L}^n(x) \right) dt \\ &= \int_0^\infty \mathcal{L}^n\Big(\{x \colon f(x) > t\}\Big) \, p t^{p-1} \, dt. \end{aligned}$$

□

THEOREM 3.4 (Hardy–Littlewood–Wiener). *There exist constants $C_p$, $p \geq 1$, such that for every $f \in L^p(\mathbb{R}^n)$ the following hold:*
(a)

$$\mathcal{L}^n\Big(\{x \in \mathbb{R}^n; Mf(x) \geq \alpha\}\Big) \leq \frac{C_1}{\alpha} \int_{\mathbb{R}^n} |f(y)|\, \mathrm{d}y, \qquad \alpha > 0,$$

(b)

$$\|Mf\|_p \leq C_p \|f\|_p \qquad \textit{for } 1 < p \leq \infty.$$

PROOF. To see that $Mf$ is measurable, consider the balls whose centers have rational coordinates and whose radii are positive rational numbers. There are countably many such balls. For each ball $\overline{B}_m$, define

$$f_m(x) := \frac{\chi_{\overline{B}_m}(x)}{\mathcal{L}^n(\overline{B}_m)} \int_{\overline{B}_m} |f|\, d\mathcal{L}^n.$$

Then $Mf(x) = \sup_m f_m(x)$ for all $x \in \mathbb{R}^n$.

(a) Clearly, it suffices to obtain an analogous estimate for the set

$$G = \{x \in \mathbb{R}^n; Mf(x) > \alpha\}.$$

Let

$$\mathcal{V} = \Big\{\overline{B}(x,r)\colon \int_{\overline{B}(x,r)} |f(y)|\, \mathrm{d}y > \alpha \mathcal{L}^n\Big(\overline{B}(x,r)\Big)\Big\}.$$

The family $\mathcal{V}$ covers $G$, and

$$\mathcal{L}^n\Big(\overline{B}(x,r)\Big) \leq \frac{1}{\alpha} \int_{\mathbb{R}^n} |f(y)|\, \mathrm{d}y$$

for every ball $\overline{B}(x,r) \in \mathcal{V}$. Thus the radii of the balls in this family are bounded. We can apply a Vitali-type covering theorem to select a pairwise disjoint subfamily $\mathcal{V}' = \{\overline{B}(x_j, r_j)\}_j$ such that

$$G \subset \bigcup_j \overline{B}(x_j, 5r_j).$$

Hence,

$$\begin{aligned}\alpha \mathcal{L}^n(G) &\leq \alpha \sum_j \mathcal{L}^n\Big(\overline{B}(x_j, 5r_j)\Big) = 5^n \alpha \sum_j \mathcal{L}^n\Big(\overline{B}(x_j, r_j)\Big) \\ &\leq 5^n \sum_j \int_{\overline{B}(x_j, r_j)} |f(y)|\, \mathrm{d}y \leq 5^n \int_{\mathbb{R}^n} |f(y)|\, \mathrm{d}y.\end{aligned}$$

This completes the proof of part (a).

(b) For $p = \infty$, the estimate is trivial with $C_\infty = 1$.

Assume, therefore, that $1 < p < \infty$. Set

$$G_t = \{x \in \mathbb{R}^n; Mf(x) > t\}, \quad \text{and} \quad f_t = f\chi_{\{|f| > t\}} \quad \text{for } t > 0.$$

If $x \in G_t$, we can find $\overline{B} \ni x$ such that

$$t\mathcal{L}^n(\overline{B}) \leq \int_{\overline{B}} |f(y)|\, \mathrm{d}y \leq \int_{\overline{B}} |f_s(y)|\, \mathrm{d}y + s\mathcal{L}^n(\overline{B}), \quad 0 \leq s \leq t,$$

or equivalently,

$$t - s \leq \fint_{\overline{B}} |f_s|.$$

Consequently,

$$Mf(x) > t \implies Mf_s(x) \geq t - s, \qquad 0 \leq s \leq t.$$

In particular,

$$Mf(x) > t \implies Mf_{t/2}(x) \geq t/2.$$

By (a), we obtain

$$\mathcal{L}^n(G_t) \le \mathcal{L}^n(\{Mf_{t/2} \ge t/2\}) \le \frac{2C_1}{t}\int_{\mathbb{R}^n} |f_{t/2}(x)|\,\mathrm{d}x = \frac{2C_1}{t}\int_{\{|f|>t/2\}} |f(x)|\,\mathrm{d}x.$$

Applying Lemma 3.3, followed by the preceding estimate and Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |Mf(x)|^p\,\mathrm{d}x &= \int_0^\infty pt^{p-1}\mathcal{L}^n(G_t)\,\mathrm{d}t\\ &\le 2pC_1\int_0^\infty \Big(\int_{\{|f|>t/2\}} t^{p-2}|f(x)|\,\mathrm{d}x\Big)\,\mathrm{d}t\\ &= 2pC_1\int_{\mathbb{R}^n}\Big(\int_0^{2|f(x)|} t^{p-2}|f(x)|\,\mathrm{d}t\Big)\,\mathrm{d}x\\ &= \frac{2pC_1}{p-1}\int_{\mathbb{R}^n}|2f(x)|^{p-1}|f(x)|\,\mathrm{d}x\\ &= \frac{2^p pC_1}{p-1}\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x,\end{aligned}$$

which proves the assertion. □

**Questions and exercises.** Throughout, let $M_c$ denote the centered Hardy–Littlewood maximal operator on $\mathbb{R}^n$:

$$M_cf(x) = \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

Question 3.5. Compute $M_c\chi_{[0,1]}$ explicitly on $\mathbb{R}$. Draw its graph and estimate

$$\mathcal{L}^1\Big(\{x : M_c\chi_{[0,1]}(x) > \lambda\}\Big), \qquad 0<\lambda<1.$$

Calculate the $L^1$ norm of $M_c\chi_{[0,1]}$.

Question 3.6. Prove that $M_c$ is positive and sublinear, and that

$$\|M_cf\|_{L^\infty} \le \|f\|_{L^\infty}.$$

Question 3.7. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Prove that $M_cf$ is lower semicontinuous. Construct an $f$ such that $M_cf$ is not continuous.

Question 3.8. Prove that

$$M_cf(x) \le Mf(x) \le 2^n M_cf(x).$$

Question 3.9. Let $\mu$ be a Borel measure on $\mathbb{R}^n$, and define

$$M_c\mu(x) := \frac{\mu(B(x,r))}{\mathcal{L}^n(B(x,r))}.$$

Can we prove an estimate for $\mu$ analogous to part (a) of the Hardy–Littlewood–Wiener theorem?

## 4. The Besicovitch covering theorem

Theorem 4.1 (Besicovitch). *For every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ with the following property. If $A \subset \mathbb{R}^n$ and $\Delta : A \to (0,\infty)$ is a bounded function, then there exist (at most countable) sets $A_1,\dots,A_N \subset A$ such that:*

- *$\{\overline{B}(x,\Delta(x)); x \in A_i\}$ is a pairwise disjoint family for every $i \in \{1,\dots,N\}$,*
- $$A \subset \bigcup_{x\in\bigcup_{i=1}^N A_i} \overline{B}(x,\Delta(x)).$$

PROOF. **The case of bounded $A$.** Set $R := \sup_A \Delta$ and choose $\overline{B}_1 := \overline{B}(a_1, r_1)$ so that $a_1 \in A$ and $r_1 := \Delta(a_1) > \frac{3}{4}R$. Suppose that we have already chosen the balls $\overline{B}_1, \dots, \overline{B}_{j-1}$, where $j \geq 2$. If

$$F_j := A \setminus \bigcup_{i=1}^{j-1} \overline{B}(a_i, r_i) = \emptyset,$$

the process stops and we set $J = j$. If $F_j \neq \emptyset$, we continue by choosing $\overline{B}_j := \overline{B}(a_j, r_j)$ such that $a_j \in F_j$ and

$$r_j := \Delta(a_j) > \frac{3}{4} \sup_{F_j} \Delta.$$

If $F_j \neq \emptyset$ for every $j$, we set $J = \infty$. In this case $\lim_{j\to\infty} r_j = 0$, since $A$ is bounded and the inequalities

$$|a_i - a_j| \geq r_i = \frac{1}{3}r_i + \frac{2}{3}r_i > \frac{1}{3}r_i + \frac{1}{2}r_j > \frac{1}{3}r_i + \frac{1}{3}r_j$$

for $i < j < J$ imply that

$$\left\{\frac{1}{3}B_j; j < J\right\} \text{ is a pairwise disjoint family.}$$

The family $\{\overline{B}_j\}_{j=1}^{J}$ covers $A$ (otherwise there would exist $a \in \bigcap_{j=1}^{\infty} F_j$ and $j_0 \in \mathbb{N}$ with $r_{j_0} \leq \frac{3}{4}\Delta(a)$, contradicting the choice of $r_{j_0}$).

Fix $k < J$. Set

$$I(k) = \{i < k : \overline{B}_i \cap \overline{B}_k \neq \emptyset\}.$$

We now prove that there exists $M \in \mathbb{N}$, depending only on $n$, that bounds $\#I(k)$ independently of $k$. To this end, we split $I(k)$ into $I_1(k)$ and $I_2(k)$ and estimate their cardinalities separately:

$$I_1(k) = \{i < k; \overline{B}_i \cap \overline{B}_k \neq \emptyset,\ r_i < 10r_k\},$$

$$I_2(k) = \{i < k; \overline{B}_i \cap \overline{B}_k \neq \emptyset,\ r_i \geq 10r_k\}.$$

**Estimate of $\#I_1(k)$.** We have $\frac{1}{3}\overline{B}_i \subset 15\overline{B}_k$ for every $i \in I_1$. Indeed, if $x \in \frac{1}{3}\overline{B}_i$, then

$$|x - a_k| \leq |x - a_i| + |a_i - a_k| \leq \frac{10}{3}r_k + r_i + r_k \leq \frac{43}{3}r_k < 15r_k.$$

Since the balls $\frac{1}{3}\overline{B}_i$ are pairwise disjoint, $I_1(k)$ has at most $60^n$ elements: for every $i \in I_1(k)$ we have (using $r_i \geq \frac{3}{4}r_k$)

$$\mathcal{L}^n\left(\frac{1}{3}\overline{B}_i\right) = \mathcal{L}^n(\overline{B}(0,1)) \cdot \left(\frac{1}{3}r_i\right)^n > \mathcal{L}^n(\overline{B}(0,1)) \cdot \left(\frac{1}{4}r_k\right)^n = \frac{1}{60^n}\mathcal{L}^n(15\overline{B}_k).$$

**Estimate of $\#I_2(k)$.** Write $b_i = a_i - a_k$. An elementary grid construction gives a family $\{Q_m; 1 \leq m \leq (22n)^n\}$ of closed cubes of side length $1/(11n)$ (so that $\operatorname{diam} Q_m \leq 1/11$) covering $[-1,1]^n$, and hence, in particular, the unit sphere. We claim that for every $1 \leq m \leq (22n)^n$ there is at most one $i \in I_2(k)$ such that $\frac{b_i}{|b_i|} \in Q_m$. This bounds the cardinality of $I_2(k)$.

If this claim were false, there would exist $i, j \in I_2(k)$, $i < j$, such that

$$\left|\frac{b_i}{|b_i|} - \frac{b_j}{|b_j|}\right| \leq \frac{1}{11}.$$

Observe that

$$r_i < |b_i| < r_i + r_k \quad \text{and} \quad r_j < |b_j| < r_j + r_k, \tag{4.1}$$

because the balls $\overline{B}_i, \overline{B}_j$ intersect $\overline{B}_k$ but do not contain $a_k$. Thus

$$\Big||b_i| - |b_j|\Big| \le |r_i - r_j| + r_k \le |r_i - r_j| + \frac{1}{10} r_j,$$

and

$$|b_j| \le r_j + r_k \le r_j + \frac{1}{10} r_j = \frac{11}{10} r_j. \tag{4.2}$$

We have

$$\begin{aligned}
|a_i - a_j| = |b_i - b_j| &\le \Big| b_i - \frac{|b_j|}{|b_i|} b_i \Big| + \Big| \frac{|b_j|}{|b_i|} b_i - b_j \Big| \\
&= \Big| \frac{|b_i|}{|b_i|} b_i - \frac{|b_j|}{|b_i|} b_i \Big| + \Big| \frac{|b_j|}{|b_i|} b_i - \frac{|b_j|}{|b_j|} b_j \Big| \\
&\le \Big| |b_i| - |b_j| \Big| + \frac{1}{11} |b_j| \\
&\le |r_i - r_j| + \frac{1}{10} r_j + \frac{1}{10} r_j \quad \text{(using (4.1) and (4.2))} \\
&\le \begin{cases} r_i - \frac{4}{5} r_j < r_i & \text{if } r_i > r_j, \\ -r_i + \frac{6}{5} r_j \le -r_i + \frac{8}{5} r_i < r_i & \text{if } r_i \le r_j. \end{cases}
\end{aligned}$$

In the second case, we used $r_j < \frac{4}{3} r_i$, since $i < j$. We have reached a contradiction, since $i < j$, and hence $a_j \notin \overline{B}_i$ (which holds if $|a_i - a_j| < r_i$). Therefore $\#I_2(k) \le (22n)^n$.

We may thus choose $M := 60^n + (22n)^n + 1$.

**Choosing $A_1, \dots, A_M$.** For every $k \in \mathbb{N}$, define $\lambda_k \in \{1, 2, \dots, M\}$ by setting $\lambda_k = k$ whenever $k \le M$. For $k > M$, define $\lambda_k$ inductively as follows: there exists $\lambda_k \in \{1, \dots, M\}$ such that

$$\overline{B}_k \cap \bigcup \{\overline{B}_i; i < k,\ \lambda_i = \lambda_k\} = \emptyset.$$

Now set $A_j = \{a_i; \lambda_i = j\}$, $j = 1, \dots, M$.

**The case of general $A$.** For each $l \in \mathbb{N}$, apply the result obtained above with $A$ replaced by

$$A^l = A \cap \{x; 3(l-1)R \le |x| < 3lR\},$$

where $R$ is the supremum of the radii. Denote the resulting sets by $A_i^l$, $i = 1, \dots, M$. Then set

$$A_i = \bigcup_{l \text{ odd}} A_i^l, \qquad A_{M+i} = \bigcup_{l \text{ even}} A_i^l, \qquad i = 1, \dots, M.$$

We have thus constructed $N := 2M$ subsets with the required properties. □

Question 4.2 (Capturing a fixed fraction of the measure). Prove that for all $n \ge 1$ there exists a $C > 0$ such that for any finite Borel measure on $\mathbb{R}^n$, $\mu$, and any Borel set $A \subset \mathbb{R}^n$ and any bounded $\Delta : A \to (0, \infty)$, there exists a finite family $\mathcal{V}$ of disjoint closed balls

$$\mu\left( A \cap \bigcup_{\overline{B} \in \mathcal{V}} \overline{B} \right) \ge \frac{1}{2N_n} \mu(A).$$

Question 4.3 (The boundedness assumption). Does the Besicovitch covering theorem remain true if $\Delta$ is not assumed to be bounded? Consider separately:

(1) bounded sets $A$;
(2) arbitrary sets $A$.

Question 4.4 (Finite versus countable coverings). If $A$ is bounded, can the selected family of balls always be chosen to be finite?

QUESTION 4.5 (Dependence on the dimension). Show that any constant $N_n$ satisfying the Besicovitch covering theorem must satisfy

$$N_n \geq 2n.$$

In particular, there is no constant valid simultaneously in all dimensions.

QUESTION 4.6 (Enlarging the selected balls). Let $\lambda > 1$. Is there a constant $C = C(n, \lambda)$ such that every pairwise disjoint family of closed balls with bounded radii satisfies

$$\sum_j \chi_{\lambda \overline{B}_j} \leq C \qquad \text{on } \mathbb{R}^n?$$

Here $\lambda \overline{B}_j$ denotes the concentric ball whose radius is multiplied by $\lambda$. If not, construct a counterexample already in $\mathbb{R}$.

QUESTION 4.7 (Replacing balls by rectangles). Does an analogue of the Besicovitch covering theorem hold in $\mathbb{R}^2$ if each prescribed ball is replaced by a closed rectangle centred at the same point, with sides parallel to the coordinate axes?

More precisely, is there a universal $N$ such that one can always select rectangles covering all their prescribed centres and partition the selected rectangles into $N$ pairwise disjoint families?

Investigate this even for finite sets of centres.

QUESTION 4.8 (From local estimates to a global estimate). Let $\mu, \nu$ be finite Borel measures on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ be Borel, and let $t > 0$. Suppose that for every $x \in A$ there exists $r_x \in (0, 1]$ such that

$$\mu(\overline{B}(x, r_x)) \leq t\, \nu(\overline{B}(x, r_x)).$$

Prove that

$$\mu(A) \leq N_n t\, \nu(\mathbb{R}^n).$$

## 5. Differentiation of measures

### 5.1. Radon measures.

DEFINITION 5.1. Let $P$ be a locally compact topological space and let $\mathcal{S}$ be a $\sigma$-algebra of subsets of $P$. We say that $\mu$ is a **positive Radon measure** on $(P, \mathcal{S})$ if:

(a) $\mathcal{S}$ contains all Borel subsets of $P$,
(b) $\mu(K) < \infty$ for every compact set $K \subset P$,
(c) $\mu(G) = \sup\{\mu(K); K \subset G \text{ is compact}\}$ for every open set $G \subset P$,
(d) $\mu(A) = \inf\{\mu(G); A \subset G,\ G \text{ is open}\}$ for every $A \in \mathcal{S}$,
(e) $\mu$ is complete.

EXAMPLE 5.2. Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable curve of finite length. Then

$$\mu(E) = \mathcal{H}^1(E \cap \Gamma), \qquad E \subset \mathbb{R}^n \text{ Borel},$$

defines a finite Borel measure. Its completion is a Radon measure, measuring the length of the part of $\Gamma$ contained in each measurable set.

QUESTION 5.3. Which of these assumptions can be omitted, and under what circumstances (for example, when $P = \mathbb{R}^n$)?

DEFINITION 5.4. Let $\mu$ be a measure on $X$. The outer measure associated with $\mu$ is defined by

$$\mu^*(A) = \inf\{\mu(E); A \subset E,\ E \text{ is } \mu\text{-measurable}\}.$$

REMARK 5.5. Let $\mu$ be a Radon measure on $(\mathbb{R}^n, \mathcal{S})$ and let $A \in \mathcal{S}$. Then there exists a Borel set $E \subset \mathbb{R}^n$ such that $A \subset E$ and $\mu(E \setminus A) = 0$. If $\nu$ is a Radon measure on $(\mathbb{R}^n, \mathcal{S}')$ with $\nu \ll \mu$, then $\mathcal{S} \subset \mathcal{S}'$.

LEMMA 5.6. *Let $\mu$ be a measure on $X$ and let $\{A_j\}_{j=1}^{\infty}$ be an increasing sequence of subsets of $X$. Then*

$$\lim_{j\to\infty} \mu^*(A_j) = \mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big).$$

PROOF. For each $j \in \mathbb{N}$, choose a $\mu$-measurable set $E_j$ such that $A_j \subset E_j$ and $\mu^*(A_j) = \mu(E_j)$. Set $M_k = \bigcup_{j=1}^{k} E_j$. Then $M_k$ is $\mu$-measurable, $A_k \subset M_k$, and $\mu(M_k) = \mu^*(A_k)$ for every $k \in \mathbb{N}$. Moreover, $\{M_k\}$ is an increasing sequence of sets. Hence

$$\lim_{k\to\infty} \mu^*(A_k) = \lim_{k\to\infty} \mu(M_k) = \mu\Big(\bigcup_{k=1}^{\infty} M_k\Big) \geq \mu^*\Big(\bigcup_{k=1}^{\infty} A_k\Big) \geq \lim_{k\to\infty} \mu^*(A_k),$$

which completes the proof. □

THEOREM 5.7 (Centred Vitali covering theorem for Radon measures). *Let $\mu$ be a Radon measure on $\mathbb{R}^n$, let $A \subset \mathbb{R}^n$ and let $\mathcal{V}$ be a family of closed balls in $\mathbb{R}^n$ with the following property:*

$$\inf\{r;\ \overline{B}(a,r) \in \mathcal{V}\} = 0 \qquad \text{for every } a \in A.$$

*Then there exists a countable pairwise disjoint family $\mathcal{G} \subset \mathcal{V}$ such that*

$$\mu\Big(A \setminus \bigcup_{\overline{B}\in\mathcal{G}} \overline{B}\Big) = 0.$$

PROOF. Suppose that the theorem holds whenever $A \subset G \subset \mathbb{R}^n$, where $G$ is a bounded open set. We show how to deduce the result on all of $\mathbb{R}^n$. Write

$$\tilde{K}_1 := \mathbb{Z} \times \mathbb{R}^{n-1}, \quad \tilde{K}_2 := \mathbb{R} \times \mathbb{Z} \times \mathbb{R}^{n-2}, \quad \ldots, \quad \tilde{K}_n := \mathbb{R}^{n-1} \times \mathbb{Z},$$

and

$$K(t) = \bigcup_{i=1}^{n} (\tilde{K}_i + te_i).$$

We find $t \in [0,1)$ such that $\mu(K(t)) = 0$ as follows. For every $m \in \mathbb{N}$, we have $\mu(Q(0,m)) < \infty$, so there are only countably many $t \in [0,1)$ such that

$$\mu\Big(Q(0,m) \cap (\tilde{K}_i + te_i)\Big) > 0.$$

Exclude these values of $t$ and repeat for every $m$. Then choose any value of $t$ that has not been excluded; we have $\mu(K(t)) = 0$. Now $\mathbb{R}^n \setminus K(t)$ is a disjoint union of countably many open cubes of side length one. Since the theorem holds in each cube separately, it follows that it holds on all of $\mathbb{R}^n$.

We therefore prove the assertion when $A$ is bounded implying that $\mu^*(A) < \infty$. Let $N$ be the natural number from the Besicovitch covering theorem, Theorem 4.1. Fix $\theta$ such that

$$1 - \frac{1}{N} < \theta < 1.$$

We claim the following. For every bounded open set $U \subset \mathbb{R}^n$, there exists a finite pairwise disjoint family $\mathcal{F} \subset \mathcal{V}$ such that

$$\bigcup_{\overline{B}\in\mathcal{F}} \overline{B} \subset U$$

and

$$\mu^*\Big((A\cap U)\setminus \bigcup_{\overline{B}\in\mathcal{F}} \overline{B}\Big) \le \theta\mu^*(A\cap U). \tag{5.1}$$

Clearly, it suffices to consider the case $\mu^*(A\cap U)>0$. For each $a\in A$ choose one $\overline{B(a,r)}\in\mathcal{V}$ such that $\overline{B(a,r)}\subset U$ and call this collection of balls $\mathcal{V}_1$. The boundedness of $r_a$ is guaranteed by the boundedness of $U$. Then by the Besicovitch covering theorem, there exist pairwise disjoint families $\mathcal{G}_1,\dots,\mathcal{G}_N\subset\mathcal{V}_1$ such that

$$A\cap U\subset \bigcup_{i=1}^{N}\bigcup_{\overline{B}\in\mathcal{G}_i}\overline{B}.$$

Consequently,

$$\mu^*(A\cap U)\le \sum_{i=1}^{N}\mu^*\Big(A\cap U\cap \bigcup_{\overline{B}\in\mathcal{G}_i}\overline{B}\Big).$$

Therefore, there exists an integer $1\le j\le N$ such that

$$\mu^*\Big(A\cap U\cap \bigcup_{\overline{B}\in\mathcal{G}_j}\overline{B}\Big)\ge \frac{1}{N}\mu^*(A\cap U)>(1-\theta)\mu^*(A\cap U).$$

Using Lemma 5.6, we find a finite family $\mathcal{F}\subset\mathcal{G}_j$ such that

$$\mu^*\Big(A\cap U\cap \bigcup_{\overline{B}\in\mathcal{F}}\overline{B}\Big)>(1-\theta)\mu^*(A\cap U).$$

The set $\bigcup_{\overline{B}\in\mathcal{F}}\overline{B}$ is $\mu$-measurable, and hence

$$\begin{aligned}\mu^*(A\cap U)&=\mu^*\Big(A\cap U\cap \bigcup_{\overline{B}\in\mathcal{F}}\overline{B}\Big)+\mu^*\Big((A\cap U)\setminus\bigcup_{\overline{B}\in\mathcal{F}}\overline{B}\Big)\\&\ge(1-\theta)\mu^*(A\cap U)+\mu^*\Big((A\cap U)\setminus\bigcup_{\overline{B}\in\mathcal{F}}\overline{B}\Big).\end{aligned}$$

This gives (5.1).

Set $U_1=A+B(0,1)$ and use (5.1) to find a finite pairwise disjoint family $\mathcal{F}_1\subset\mathcal{V}$ such that

$$\bigcup_{\overline{B}\in\mathcal{F}_1}\overline{B}\subset U_1 \qquad\text{and}\qquad \mu^*\Big((A\cap U_1)\setminus\bigcup_{\overline{B}\in\mathcal{F}_1}\overline{B}\Big)\le\theta\mu^*(A\cap U_1).$$

Continuing inductively, we obtain a sequence of open sets $(U_j)$ and finite pairwise disjoint families $(\mathcal{F}_j)$ such that

$$U_{j+1}=U_j\setminus\bigcup_{\overline{B}\in\mathcal{F}_j}\overline{B},\qquad \mathcal{F}_j\subset\mathcal{V},\qquad \bigcup_{\overline{B}\in\mathcal{F}_j}\overline{B}\subset U_j,$$

and

$$\mu^*(A\cap U_{j+1})=\mu^*\Big((A\cap U_j)\setminus\bigcup_{\overline{B}\in\mathcal{F}_j}\overline{B}\Big)\le\theta\mu^*(A\cap U_j)$$

for every $j\in\mathbb{N}$. Thus $\mu^*(A\cap U_{j+1})\le\theta^j\mu^*(A)$ for every $j\in\mathbb{N}$. Since $\mu^*(A)<\infty$, it follows that

$$\mu^*\Big(A\setminus\bigcup_{j=1}^{\infty}\bigcup_{\overline{B}\in\mathcal{F}_j}\overline{B}\Big)=0.$$

We therefore set $\mathcal{G}=\bigcup_{j=1}^{\infty}\mathcal{F}_j$, completing the proof. □

Question 5.8. Where in the previous proof was the infinitesimal radius assumption for balls in $\mathcal{V}$ used?

### 5.2. Differentiating measures.

DEFINITION 5.9. Let $\nu$ and $\mu$ be measures on $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We define:

- the *upper derivative* of $\nu$ with respect to $\mu$ at $x$ by
$$\overline{D}(\nu,\mu,x) = \limsup_{r\to 0+} \frac{\nu\Big(\overline{B(x,r)}\Big)}{\mu\Big(\overline{B(x,r)}\Big)},$$
provided that the expression on the right is defined;
- the *lower derivative* of $\nu$ with respect to $\mu$ at $x$ by
$$\underline{D}(\nu,\mu,x) = \liminf_{r\to 0+} \frac{\nu\Big(\overline{B(x,r)}\Big)}{\mu\Big(\overline{B(x,r)}\Big)},$$
provided that the expression on the right is defined;
- the *derivative* of $\nu$ with respect to $\mu$ at $x$, denoted by $D(\nu,\mu,x)$, as the common value of $\overline{D}(\nu,\mu,x)$ and $\underline{D}(\nu,\mu,x)$, whenever this common value is defined.

REMARK 5.10. Let $\mu$ and $\nu$ be Radon measures on $\mathbb{R}^n$. The value $\overline{D}(\nu,\mu,x)$ (or $\underline{D}(\nu,\mu,x)$) is well defined if and only if $\mu(\overline{B(x,r)}) > 0$ for every $r > 0$.

THEOREM 5.11. *Let $\nu$ and $\mu$ be Radon measures on $\mathbb{R}^n$. Then $\overline{D}(\nu,\mu,x)$ and $\underline{D}(\nu,\mu,x)$ are defined $\mu$-almost everywhere.*

PROOF. Write
$$M = \{x \in \mathbb{R}^n; \overline{D}(\nu,\mu,x) \text{ is not defined}\}.$$
Then $M$ consists of the points $x$ for which there exists $r_x > 0$ such that $\mu(B(x,r_x)) = 0$. Let $\mathcal{V}$ be the family of all closed balls $\overline{B}$ such that $\mu(\overline{B}) = 0$. Then $\mathcal{V}$ is a centred Vitali covering of $M$. By Theorem 5.7, there exists a countable pairwise disjoint family
$$\{\overline{B}_1, \overline{B}_2, \dots\} \subset \mathcal{V}$$
such that
$$\mu\Big(M \setminus \bigcup_i \overline{B}_i\Big) = 0.$$
Since
$$\mu\Big(\bigcup_i \overline{B}_i\Big) = \sum_{i=1}^{\infty} \mu(\overline{B}_i) = 0,$$
we obtain $\mu(M) = 0$. The proof for $\underline{D}(\nu,\mu,x)$ is analogous. □

THEOREM 5.12. *Let $\nu$ and $\mu$ be Radon measures on $\mathbb{R}^n$, let $c \in (0,\infty)$, and let $M \subset \mathbb{R}^n$.*

(i) *If $\overline{D}(\nu,\mu,x) > c$ for every $x \in M$, then $\nu^*(M) \geq c\mu^*(M)$.*
(ii) *If $\underline{D}(\nu,\mu,x) < c$ for every $x \in M$, then $\nu^*(M) \leq c\mu^*(M)$.*

PROOF. (i) Fix $\varepsilon > 0$. There exists an open set $G \subset \mathbb{R}^n$ such that $M \subset G$ and $\nu(G) \leq \nu^*(M) + \varepsilon$. Set
$$\mathcal{V} = \{\overline{B} \in \mathcal{B}; \overline{B} \subset G,\ \nu(\overline{B}) > c\mu(\overline{B})\}.$$
The family $\mathcal{V}$ is a Vitali covering of $M$. Therefore, there exists a countable pairwise disjoint subfamily $\{\overline{B}_1, \overline{B}_2, \dots\} \subset \mathcal{V}$ such that
$$\mu\Big(M \setminus \bigcup_i \overline{B}_i\Big) = 0.$$

Therefore,

$$\nu^*(M)+\varepsilon \geq \nu(G) \geq \nu\Big(\bigcup_i \overline{B}_i\Big) = \sum_i \nu(\overline{B}_i)$$
$$\geq c\sum_i \mu(\overline{B}_i) = c\mu\Big(\bigcup_i \overline{B}_i\Big) \geq c\mu^*(M).$$

Letting $\varepsilon \to 0+$ gives the desired inequality.

(ii) Fix $k \in \mathbb{N}$. There exists an open set $G_k \subset \mathbb{R}^n$ such that $M \subset G_k$ and $\mu(G_k) \leq \mu^*(M) + 1/k$. Set

$$\mathcal{V}_k = \{\overline{B} \in \mathcal{B}; \overline{B} \subset G_k,\ \nu(\overline{B}) < c\mu(\overline{B})\}.$$

The family $\mathcal{V}_k$ is a Vitali covering of $M$. Because $\mu+\nu$ is a Radon measure, there exists a countable pairwise disjoint subfamily $\{\overline{B}_1^k, \overline{B}_2^k, \dots\} \subset \mathcal{V}_k$ such that

$$\nu\Big(M \setminus \bigcup_i \overline{B}_i^k\Big) = \mu\Big(M \setminus \bigcup_i \overline{B}_i^k\Big) = 0.$$

It holds that

$$\nu^*(M) \leq \nu\Big(\bigcup_i \overline{B}_i^k\Big) = \sum_i \nu(\overline{B}_i^k) \leq c\sum_i \mu(\overline{B}_i^k)$$
$$= c\mu\Big(\bigcup_i \overline{B}_i^k\Big) \leq c\mu(G_k) \leq c\Big(\mu^*(M) + \frac{1}{k}\Big).$$

It follows that

$$\nu^*(M) \leq c\mu^*(M).$$

□

THEOREM 5.13. *Let $\nu$ and $\mu$ be Radon measures on $\mathbb{R}^n$. Then $D(\nu,\mu,x)$ exists and is finite $\mu$-almost everywhere.*

PROOF. Write

$$D := \{x \in \mathbb{R}^n; D(\nu,\mu,x) \in [0,\infty)\},$$
$$N_1 := \{x \in \mathbb{R}^n; \overline{D}(\nu,\mu,x) \text{ or } \underline{D}(\nu,\mu,x) \text{ is not defined}\},$$
$$N_3 := \{x \in \mathbb{R}^n; \overline{D}(\nu,\mu,x) = \infty\},$$
$$N_4 := \{x \in \mathbb{R}^n; \underline{D}(\nu,\mu,x) < \overline{D}(\nu,\mu,x)\}.$$

Then

- $D = \mathbb{R}^n \setminus (N_1 \cup N_3 \cup N_4)$,
- $\mu(N_1) = 0$.

Further, define

$$A_k = \{x \in \mathbb{R}^n; \overline{D}(\nu,\mu,x) > k\},$$

and

$$A(r,s) = \{x \in \mathbb{R}^n; \underline{D}(\nu,\mu,x) < s < r < \overline{D}(\nu,\mu,x)\}, \qquad s,r \in \mathbb{Q}^+,\ s<r.$$

Then

$$N_3 = \bigcap_{k=1}^{\infty} A_k, \qquad N_4 = \bigcup\{A(r,s); r,s \in \mathbb{Q}^+,\ s<r\}.$$

We show that $\mu(N_3) = 0$, which also gives measurability automatically. Let $E \subset N_3$ be bounded. By Theorem 5.12 (i),

$$k\mu^*(E) \leq \nu^*(E) < \infty \qquad \text{for every } k \in \mathbb{N}.$$

Hence $\mu^*(E) = 0$, and consequently $\mu(N_3) = 0$, since $N_3$ is a countable union of bounded sets.

We now show that $\mu(N_4) = 0$. It suffices to prove that $\mu(A(r,s)) = 0$ for every $s, r \in \mathbb{Q}^+$ with $s < r$. Let $E \subset A(r,s)$ be bounded. By Theorem 5.12 (ii), there exists $H \subset E$ such that $\mu(E \setminus H) = 0$ and $\nu^*(H) \leq s\mu^*(E)$. By Theorem 5.12 (i), $r\mu^*(H) \leq \nu^*(H)$. We therefore obtain

$$r\mu^*(E) = r\mu^*(H) \leq \nu^*(H) \leq s\mu^*(E) < \infty.$$

Since $r > s > 0$, it follows that $\mu^*(E) = 0$. This implies $\mu(A(r,s)) = 0$. □

Question 5.14. Consider the following measures on $\mathbb{R}$, completed on their respective measurable sets:

$$\mu = \sum_{k=1}^{\infty} \delta_k, \qquad \nu = \sum_{k=1}^{\infty} 2^{-k}\delta_{1/k}, \qquad \eta = \sum_{k=1}^{\infty} \frac{1}{k}\delta_{1/k}.$$

(1) Which measures are globally finite? Which are locally finite? Which are Radon?
(2) All three measures assign finite mass to every singleton. Does this imply local finiteness?
(3) Compute $\nu(\{0\})$ and determine whether $0 \in \operatorname{supp}\nu$. Must every point in the support of an atomic measure be an atom?
(4) Is the restriction of $\eta$ to the space $(0,\infty)$ a Radon measure? Explain why removing one point changes the answer.

Question 5.15. Let $\mu, \nu$ be Radon measures on $\mathbb{R}^n$, and suppose that $\mu(\{x\}) > 0$. Prove that

$$D(\nu, \mu, x) = \frac{\nu(\{x\})}{\mu(\{x\})}.$$

Question 5.16. Let $\mu$ be a Radon measure on $\mathbb{R}^n$, let $f : \mathbb{R}^n \to [0,\infty)$ be continuous, and define

$$\nu(E) = \int_E f \, d\mu.$$

By direct calculation prove that

$$D(\nu, \mu, x) = f(x) \quad \text{for every } x \in \operatorname{supp}\mu.$$

Question 5.17. Find a pair of Radon measures $\mu, \nu$ such that

(1) for some $E$ Borel set it holds that

$$\nu(E) \neq \int_E D(\nu, \mu, x) \, d\mu(x)$$

(2) Does the Radon–Nikodym derivative $d\nu/d\mu$ exist?
(3) Find the Lebesgue decomposition of $\nu$ with respect to $\mu$. Which part of $\nu$ is recovered by integrating $D(\nu, \mu, \cdot)$?
(4) What happens when you swap the role of $\mu$ and $\nu$