(24) $\int x^2 \sin^2 x \, dx = \int x^2 \left(\frac{1-\cos 2x}{2}\right) dx$

$\sin^2 y = \frac{1}{2}(1-\cos 2y)$

$\cos^2 y = \frac{1}{2}(1+\cos 2y)$

$$= \underbrace{\int \frac{x^2}{2} dx}_{\frac{x^3}{6}} + \underbrace{\int x^2 \left(-\frac{\cos 2x}{2}\right) dx}_{I_1}$$

$$I_1 = \int \underbrace{\frac{x^2}{2}}_{u} \underbrace{(-\cos 2x)}_{v'} dx = -\frac{x^2}{4}\sin 2x + \frac{1}{2}\underbrace{\int x \sin 2x \, dx}_{I_2}$$

$u' = x \; ; \; v = -\frac{\sin 2x}{2}$

$$I_2 = \int \underbrace{x}_{u} \underbrace{\sin 2x}_{v'} \, dx = -\frac{x}{2}\cos 2x + \frac{1}{2}\underbrace{\int \cos 2x \, dx}_{\frac{\sin 2x}{2}}$$

$u' = 1 \; , \; v = -\frac{1}{2}\cos 2x$

celkem: $\int x^2 \sin^2 x \, dx = \frac{x^3}{6} - \frac{x^2}{4}\sin 2x - \frac{x}{4}\cos 2x + \frac{\cos 2x}{8}$

v $\mathbb{R}$. [m.x.]

(25) $$\int \underbrace{x}_{u}\cdot\underbrace{\frac{1}{\cos^2 x}}_{v'}\,dx = x\cdot \operatorname{tg} x - \underbrace{\int \operatorname{tg} x\,dx}_{I_1}.$$

$u' = 1$; $v = \operatorname{tg} x$

$$I_1 = \int \frac{\sin x}{\cos x}\,dx \;\left|\; \begin{matrix} y = \cos x \\ dy = -\sin x\,dx \end{matrix} \right\} = -\int \frac{dy}{y} = -\ln|y| = -\ln|\cos x|$$

kde? : v intervalech, kde $y \neq 0$ tj: $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$. $k \in \mathbb{Z}$

$$\int \frac{x}{\cos^2 x}\,dx = x \operatorname{tg} x + \ln|\cos x| \; ; \; \text{v intervalech}$$

[m.x.]

(26)

$$\int \underbrace{2\cdot \operatorname{arctg}\frac{x}{2}}\,dx = 2x \operatorname{arctg}\frac{x}{2} - \underbrace{\int \frac{x}{1+\left(\frac{x}{2}\right)^2}\,dx}_{I_1}$$

$u' = 1 \quad v$

$u = x \quad v' = \dfrac{1}{1+\left(\frac{x}{2}\right)^2}$

$$I_1 = 2\int \frac{x/2}{1+(x/2)^2}\,dx \;\left|\; \begin{matrix} (x/2)^2 + 1 = y \\ x/2\,dx = dy \end{matrix} \right\} = 2\int \frac{dy}{y}$$

$$= 2\ln|y| = 2\ln\left(1+(x/2)^2\right) \text{ v } \mathbb{R}.$$

celkem

$$2x \operatorname{arctg}\frac{x}{2} - 2\ln\left(1+\frac{x^2}{4}\right) \text{ v } \mathbb{R}.$$

[m.x.]

(27) $\int \underbrace{\arcsin x}_{} \, dx = x \cdot \arcsin x - \underbrace{\int \frac{x}{\sqrt{1-x^2}} \, dx}_{I_1}$

$u' = 1 \quad v$

$u = x \quad v' = \frac{1}{\sqrt{1-x^2}}$

$I_1 = \frac{1}{2} \int \frac{2x}{\sqrt{1-x^2}} \, dx \; \left| \begin{array}{l} y = 1-x^2 \\ dy = -2x\,dx \end{array} \right\} = -\int \frac{dy}{2\sqrt{y}} = -\sqrt{y} = -\sqrt{1-x^2}$

kde? v intervalech, kde $y > 0$; tj. $x \in (-1,1)$.

celkem: $\int \arcsin x \, dx = x \cdot \arcsin x + \sqrt{1-x^2}$ v $(-1,1)$.
[m. x.]

(28) $\int \underbrace{\frac{1}{x^2}}_{u'} \cdot \underbrace{\arccos x}_{v} \, dx = -\frac{\arccos x}{x} - \underbrace{\int \frac{dx}{x\sqrt{1-x^2}}}_{I_1}$.

$u = -\frac{1}{x} \quad v' = \frac{-1}{\sqrt{1-x^2}}$

$I_1 = \frac{1}{2} \int \frac{-2x \, dx}{x^2\sqrt{1-x^2}} \; \left| \begin{array}{l} y = 1-x^2 \\ dy = -2x\,dx \end{array} \right\} = \frac{1}{2} \int \frac{dy}{(1-y)\sqrt{y}} \; \left| \begin{array}{l} \sqrt{y} = t \\ y = t^2 \\ dy = 2t\,dt \end{array} \right.$

$= \int \frac{t \, dt}{(1-t^2) \cdot t} = \int \frac{dt}{1-t^2} = \frac{1}{2} \int \frac{1}{1-t} + \frac{1}{1+t} \, dt$

$= \frac{1}{2} \ln \left| \frac{1+t}{1-t} \right|$.

---

celkem: $\ldots = -\frac{\arccos x}{x^2} + \frac{1}{2} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}$;

kde?: $x \in (-1,0)$

$x \in (0,1)$.

(29) $\int \underbrace{\ln\left(\sqrt{x+1}-\sqrt{x-1}\right)} dx$; $\mathcal{D}=?$ $x+1>0$ & $x-1>0$

$\boxed{x>1}$

$u'=1 \quad v$ $\sqrt{x+1}>\sqrt{x-1}$ o.k.

$$u=x \quad v'=\frac{\frac{1}{2\sqrt{x+1}}-\frac{1}{2\sqrt{x-1}}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{1}{2\sqrt{x^2-1}}\cdot\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x+1}-\sqrt{x-1}}$$

$$=-\frac{1}{2\sqrt{x^2-1}} \quad [g.m.]$$

per-partes: $x\cdot\ln\left(\sqrt{x+1}-\sqrt{x-1}\right)+\underbrace{\int\frac{x}{2\sqrt{x^2-1}}dx}_{I_1}$

$$I_1=\frac{1}{4}\int\frac{2x\,dx}{\sqrt{x^2-1}}\ \left|\begin{array}{l} y=x^2-1\\ dy=2x\,dx\end{array}\right\}=\frac{1}{4}\int\frac{dy}{\sqrt{y}}=\frac{1}{2}\sqrt{y}$$

$$=\frac{1}{2}\sqrt{x^2-1} \ \text{(m.x.)}$$

celkem:

$$=x\cdot\ln\left(\sqrt{x+1}-\sqrt{x-1}\right)+\frac{1}{2}\sqrt{x^2-1}\ ,\ v\ (1,+\infty).$$

(30) $\int\frac{\ln \operatorname{arctg} x}{1+x^2}dx\ \left|\begin{array}{l} y=\operatorname{arctg}x\\ dy=\frac{dx}{1+x^2}\end{array}\right\}=\int \ln y\,dy$

per-partes: $\int \underbrace{1}_{u'}\cdot\underbrace{\ln y}_{v}\,dy=y\ln y-\int dy=y(\ln y-1)$

$v\ (0,+\infty).$

$u=y \quad v'=\frac{1}{y}$

celkem: $=\operatorname{arctg}x\cdot\left(\ln\operatorname{arctg}x-1\right)$; $x\in(0,+\infty)$.

[m.x.]

(31) $\displaystyle\int \frac{\operatorname{tg}(1/x)}{x^2}\,dx \;\left|\begin{array}{l} y=\frac{1}{x} \\ dy=-\frac{1}{x^2}dx \end{array}\right\} = -\int \operatorname{tg} y\, dy$

$\displaystyle = \int \frac{-\sin y\, dy}{\cos y} \;\left|\begin{array}{l} z=\cos y \\ dz=-\sin y\, dy \end{array}\right\} = \int \frac{dz}{z} = \ln|z|\,;$

kde platí? $z \in (-\infty, 0)$ nebo $(0, +\infty)$

$\cos y \neq 0: \quad y \in \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right); \; k \in \mathbb{Z}$

$x \in \left(-\frac{2}{\pi}, 0\right)$ nebo $\left(0, \frac{2}{\pi}\right)$

$= \ln\left|\cos\frac{1}{x}\right|$ nebo $\left(\left(-\frac{\pi}{2}+k\pi\right)^{-1}, \left(\frac{\pi}{2}+k\pi\right)^{-1}\right), \; k \in \mathbb{Z}\setminus\{0\}.$

(32) $\displaystyle\int \frac{dx}{\cos^3 x} = \int \frac{\cos x\, dx}{\cos^4 x} \;\left|\begin{array}{l} y=\sin x\,; \quad \cos^2 x = 1-y^2 \\ dy=\cos x\, dx \end{array}\right.$

$\displaystyle = \int \frac{dy}{(1-y^2)^2} = I.$ $\qquad y \in (-\infty,-1), (-1,1), (1,+\infty).$

tak: $\displaystyle J = \int \frac{dy}{1-y^2} = \frac{1}{2}\int \frac{1}{1-y} + \frac{1}{1+y}\,dy = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|.$

per-partes: $\displaystyle J = \int 1\cdot\frac{1}{1-y^2}\,dy = \frac{y}{1-y^2} - 2\int\frac{y^2}{(1-y^2)^2} = \frac{y}{1-y^2} + 2(J-I)$

$\displaystyle 2I = \frac{y}{1-y^2} + J\,;$

$\displaystyle I = \frac{+y}{2(1-y^2)} + \frac{1}{4}\ln\left|\frac{1+y}{1-y}\right|_{\text{m.x.}}\,; \quad y$ výše.

celkem: $\displaystyle = \frac{\sin x}{2\cos^2 x} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$ $\qquad x \in \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$ $k \in \mathbb{Z}.$

(33) $\int \frac{x^2\,dx}{(8x^3+27)^{2/3}}$ $\quad \mathcal{D} = ?$ $\quad 8x^3 > -27$

$x^3 > (3/2)^3$

$x \in (3/2, +\infty).$

$y = 8x^3 + 27$

$dy = 24x^2\,dx$ $\quad = \frac{1}{24} \int \frac{dy}{(y)^{2/3}} = \frac{1}{24} \int y^{-2/3}\,dy$

$= \frac{1}{24} \cdot 3\, y^{1/3} = \frac{1}{8} (8x^3 + 27)^{1/3}$ [n.x.]

(34) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\,dx \left| \begin{array}{l} y = \sin x - \cos x \\ dy = \cos x + \sin x\,dx \end{array} \right\} = \int \frac{dy}{\sqrt[3]{y}}$

$= \frac{3}{2} \sqrt[3]{y^2}$; $\quad y \in (-\infty, 0)$ nebo $(0, +\infty)$.

$\sin x = \cos x : \ \operatorname{tg} x = 1 : x = \frac{\pi}{4}.$

celkem: $\quad = \frac{3}{2} \sqrt[3]{(\sin x - \cos x)^2}$; $\quad x \in \left(\frac{\pi}{4} + k\pi, \frac{5\pi}{4} + k\pi\right).$

(35) $\int \frac{dx}{\sqrt{e^x - 1}}$ $\left| \begin{array}{l} y = \sqrt{e^x - 1} \\ y^2 = e^x - 1 \\ x = \ln(y^2 + 1) \\ dx = \frac{2y}{y^2+1}\,dy \end{array} \right\} = \int \frac{2\cancel{y}\,dy}{\cancel{y}(y^2+1)}$

$x \in (0, +\infty).$

$= 2 \operatorname{arctg} y.$

$y \in \mathbb{R}.$

celkem:

$= 2 \operatorname{arctg} \sqrt{e^x - 1}$; $\quad x \in (0, +\infty).$

M.X.

(36) $\int \frac{dx}{2+\sin x}$ | $t = \operatorname{tg}\frac{x}{2}$; $x \in (-\pi, \pi) \longleftrightarrow t \in \mathbb{R}$

$\sin x = \frac{2t}{1+t^2}$ $\quad dx = \frac{2}{1+t^2}\,dt$

$$= \int \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{dt}{t^2+t+1} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)$$

$t \in \mathbb{R}$.

oznaczmy $F_0(x) = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2\operatorname{tg}\frac{x}{2}+1}{\sqrt{3}}\right)$; x

stąd: $\int \frac{dx}{2+\sin x} = F_0(x)$; $x \in (-\pi, \pi)$; ogólniej

dla $x \in ((2k-1)\pi, (2k+1)\pi)$

$\forall k \in \mathbb{Z}$.

zauważmy: $\lim_{x \to \pi-} F_0(x) = \frac{\pi}{\sqrt{3}}$; $\lim_{x \to -\pi+} F_0(x) = -\frac{\pi}{\sqrt{3}}$.

połóż:
$$F_1(x) = \begin{cases} F_0(x); & x \in (-\pi, \pi) \\ \frac{\pi}{\sqrt{3}}; & x = \pi \\ F_0(x) + \frac{2\pi}{\sqrt{3}}; & x \in (\pi, 3\pi). \end{cases}$$

jest $F_1(x)$ ciągła w $(-\pi, 3\pi)$;

zatem $\int \frac{dx}{2+\sin x} = F_1(x)$ w $(-\pi, 3\pi)$.

(37) $$\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{\sin x \, dx}{(2+\cos x)\sin^2 x} \quad \left| \begin{array}{l} y = \cos x \\ dy = -\sin x \, dx \\ \sin^2 x = 1-\cos^2 x = 1-y^2. \end{array} \right.$$

$$= \int \frac{dy}{(y+2)(y^2-1)} = \int \frac{\frac{1}{3}}{y+2} - \frac{\frac{1}{2}}{y+1} + \frac{1/6}{y-1}$$

$$= \frac{1}{3} \ln|y+2| - \frac{1}{2} \ln|y+1| + \frac{1}{6} \ln|y-1| \quad ; \quad y \in (-\infty,-2),\ (-2,-1),\ (-1,1),\ (1,+\infty)$$

$$= \frac{1}{3} \ln(\cos x + 2) - \frac{1}{2} \ln(\cos x + 1) + \frac{1}{6} \ln(1-\cos x)$$

$$x \in (k\pi, (k+1)\pi).$$

„standardní postup": $t = \operatorname{tg} \frac{x}{2}$; ... vede na $\int \frac{t^2+1}{t(t^2+3)} dt$

(38) $$\int \frac{\sin x \cos x}{\sin x + \cos x} dx \quad \left| \begin{array}{ll} t = \operatorname{tg} \frac{x}{2} ; & \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2}{1+t^2} dt & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right.$$

$$= \int \frac{\frac{2t}{1+t^2} \cdot \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{4t(1-t^2)}{(t^2+1)^2(2t+1-t^2)} dt = \dots$$

$$\frac{4t(t^2-1)}{(t^2+1)^2(t^2-2t-1)} = \frac{A}{t-\sqrt{2}-1} + \frac{B}{t+\sqrt{2}-1} + \frac{Ct+D}{t^2+1} + \frac{Et+F}{(t^2+1)^2}$$

$$= \frac{1}{2\sqrt{2}} \left(\frac{1}{t-\sqrt{2}-1} - \frac{1}{t+\sqrt{2}-1} \right) - \frac{1}{t^2+1} + \frac{2t+2}{(t^2+1)^2}$$

$$\dots = \frac{1}{2\sqrt{2}} \ln \left| \frac{t-\sqrt{2}-1}{t+\sqrt{2}-1} \right| - \operatorname{arctg} t + \underbrace{\int \frac{2t+2}{(t^2+1)^2} dt}_{I}.$$

38 dokončení: $I = \int \frac{2t}{(t^2+1)^2}\,dt + 2\underbrace{\int \frac{dt}{(t^2+1)^2}}_{I_2} \doteq I_1 + 2I_2$

$I_1 = \left| \begin{array}{l} t^2 = u \\ 2t\,dt = du \end{array} \right\} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1} = -\frac{1}{t^2+1}$;

$I_2 = \left| \begin{array}{l} \text{viz indukční} \\ \text{vzorec; př. 48} \end{array} \right) = \frac{t}{2(t^2+1)} + \frac{1}{2}\cdot \operatorname{arctg} t$.

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celkem: $I = \frac{-1}{t^2+1} + \frac{t}{t^2+1} + \operatorname{arctg} t$;

a tedy: $\int \frac{4t(t^2-1)\,dt}{(t^2+1)^2(t^2-2t-1)} = \frac{1}{2\sqrt{2}} \ln\left|\frac{t-\sqrt{2}-1}{t+\sqrt{2}-1}\right| + \frac{t-1}{t^2+1} = G(t)$

$t \in (-\infty, 1-\sqrt{2}), (1-\sqrt{2}, 1+\sqrt{2})$
$(1+\sqrt{2}, +\infty)$.

$F(x) = G(\operatorname{tg}\frac{x}{2})$ je p.f. k $\frac{\sin x\cdot\cos x}{\sin x+\cos x}$ kde?

vyloučit: $\sin x = -\cos x \iff \operatorname{tg}\frac{x}{2} = 1 \pm \sqrt{2}$

$x = -\frac{\pi}{4} + k\pi$;

tj.: $\int \frac{\sin x \cos x}{\sin x + \cos x}\,dx = F(x)$ v $(-\pi, -\frac{\pi}{4})$
$(-\frac{\pi}{4}, \frac{3\pi}{4})$
$(\frac{3\pi}{4}, \pi)$.

protože $\lim\limits_{\substack{x\to\pi- \\ -\pi+}} F(x) = 0$, lze $F(x)$ dodefinovat zde 0
spojitě a mít p.f. na
intervalech: $(-\frac{\pi}{4}+k\pi, \frac{3\pi}{4}+k\pi)$; $k \in \mathbb{Z}$.

(39) $\displaystyle\int \frac{\cos x \, dx}{2\sin x - 3\cos x + 6}$, $\left|\; \begin{array}{ll} t = \operatorname{tg}\frac{x}{2} & \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2\,dt}{1+t^2} & \cos x = \frac{1-t^2}{1+t^2} \end{array}\right.$

$$= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{4t}{1+t^2} - \frac{3(1-t^2)}{1+t^2} + 6} \cdot \frac{2\,dt}{1+t^2} = \int \frac{2(1-t^2)\,dt}{(t^2+1)(9t^2+4t+3)} \,.$$

$$= \frac{1}{13}\int \frac{36t+44}{9t^2+4t+3}\,dt - \frac{1}{13}\int \frac{4t+6}{t^2+1}\,dt = \frac{1}{13}\left(I_1 - I_2\right).$$

$$I_2 = 2\ln(t^2+1) + 6\operatorname{arctg} t\,, \quad t \in (-\infty, +\infty).$$

$$I_1 = 2\underbrace{\int \frac{18t+4}{9t^2+4t+3}\,dt}_{\ln(9t^2+4t+3)} + 36\underbrace{\int \frac{dt}{9t^2+4t+3}}_{I_3}$$

ad I_3: $\quad 9t^2+4t+3 = \left(3t+\frac{2}{3}\right)^2 + \frac{23}{9} = \frac{23}{9}\left[\left(\frac{9t+2}{\sqrt{23}}\right)^2 + 1\right]$

$$I_3 = \frac{9}{23}\cdot\int \frac{dt}{\left(\frac{9t+2}{\sqrt{23}}\right)^2+1} = \underbrace{\frac{9}{23}\cdot\frac{\sqrt{23}}{9}}_{\frac{1}{\sqrt{23}}} \operatorname{arctg}\left(\frac{9t+2}{\sqrt{23}}\right)$$

m.x.

celkem: $\displaystyle = \frac{1}{13}\left\{ 2\ln\frac{(9t^2+4t+3)}{t^2+1} + \frac{1}{\sqrt{23}}\operatorname{arctg}\left(\frac{9t+2}{\sqrt{23}}\right) - 6\operatorname{arctg} t \right\}$

$\to$ dosadit $t = \operatorname{tg}\frac{x}{2}$. $\rightsquigarrow F(x)$; $x \in (-\pi, \pi)$ $\qquad t \in \mathbb{R}$.

$$\lim_{x\to\pi-} F(x) = \frac{1}{13}\left\{2\ln 9 + \frac{\pi}{2}\left(\frac{1}{\sqrt{23}} - 6\right)\right\}.$$

$-\pi+$ $\qquad -$

lze napojit!!

(40) $\int \frac{dx}{\sin x}$; $\left| \; t = \operatorname{tg}\frac{x}{2} \; \ldots \right\} = \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$

$= \int \frac{dt}{t} = \ln|t| = \ln|\operatorname{tg}\frac{x}{2}|$; $\quad t \in (-\infty, 0), (0, +\infty)$

$\quad \ldots \; x \in (k\pi, (k+1)\pi)$.

---

(41) $\int \frac{\sin^2 x}{1+\sin^2 x}\, dx \quad \left| \begin{array}{l} t = \operatorname{tg} x \\ dx = \frac{dt}{1+t^2} \end{array} \right.$; $\quad \sin x = \frac{t}{\sqrt{1+t^2}}$, $\quad \cos x = \frac{1}{\sqrt{1+t^2}}$

$= \int \frac{\frac{t^2}{1+t^2}}{1+\frac{t^2}{1+t^2}} \cdot \frac{dt}{1+t^2} = \int \frac{t^2\, dt}{(2t^2+1)(t^2+1)}$. m.x.

$= \int \frac{1}{t^2+1} - \frac{1}{2t^2+1}\, dt = \operatorname{arctg} t - \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\, t)$;

$t \in (-\infty, \infty) \; \ldots \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

označení: $F_0(x) := \operatorname{arctg}\left(\operatorname{tg}\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\sqrt{2}\, \operatorname{tg}\frac{x}{2}\right)$;

$f(x) = \frac{\sin^2 x}{1+\sin^2 x}$; zřejmě: $F_0'(x) = f(x)$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

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definuji: $F_1(x) := \begin{cases} F_0(x); & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right); & x = \frac{\pi}{2} \\ F_0(x) + \pi\left(1 - \frac{1}{\sqrt{2}}\right); & x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$

$\Rightarrow F_1'(x) = f(x)$; $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

POZOR: $\operatorname{arctg}\left(\operatorname{tg}\frac{x}{2}\right) \neq x$ mimo $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(42) $\displaystyle\int \mathrm{tg}^5 x\, dx \;\left|\begin{array}{l} t = \mathrm{tg}\, x \\ dx = \dfrac{dt}{1+t^2} \end{array}\right\} = \int \frac{t^5}{1+t^2}\, dt$ $\qquad \sin^2 x + \cos^2 x = 1$

$\displaystyle = \int t^3 - t + \frac{t}{t^2+1}\, dt = \frac{t^4}{4} - \frac{t^2}{2} + \ln\sqrt{t^2+1}$; $\quad \mathrm{tg}^2 x + 1 = \dfrac{1}{\cos^2 x}$

$\cos x \neq 0.$

m.x.

$\displaystyle = \frac{1}{4}\,\mathrm{tg}^4 x - \frac{1}{2}\,\mathrm{tg}^2 x - \ln|\cos x|$; $\quad x \in \left((k-\tfrac{1}{2})\pi, (k+\tfrac{1}{2})\pi\right).$

(43) $\displaystyle\int \frac{1-\mathrm{tg}\, x}{1+\mathrm{tg}\, x}\, dx = \left|\begin{array}{l} t = \mathrm{tg}\, x \\ dx = \dfrac{dt}{1+t^2} \end{array}\right\} = \int \frac{1-t}{(1+t)(1+t^2)}\, dt$

$\displaystyle = \int \frac{1}{t+1} - \frac{t}{t^2+1}\, dt = \ln\left|\frac{t+1}{\sqrt{t^2+1}}\right|$; $\quad t \in (-\infty, -1)$, $(1, +\infty)$.

m.x.

$\displaystyle f(t) = \frac{1-\mathrm{tg}\, x}{1+\mathrm{tg}\, x}$; $\displaystyle F_0(t) = \ln\left|\frac{\mathrm{tg}\, x + 1}{\sqrt{\mathrm{tg}^2 x + 1}}\right| = \ln|\cos x \cdot (\mathrm{tg}\, x + 1)|$

$x \neq -\frac{\pi}{4}$ $(\mathrm{tg}\, x \neq -1)$. $\qquad = \ln|\cos x + \sin x|$,

$F_0'(t) = f(t)$; platí dokonce i pro $x = \pm\frac{\pi}{2}$.

pokud napíše rozšíříme: $\displaystyle f(t) = \frac{\cos x - \sin x}{\cos x + \sin x}$

(44) $\displaystyle\int \frac{1+\mathrm{tg}^2 x}{2+\mathrm{tg}\, x}\, dx = \left|\begin{array}{l} t = \mathrm{tg}\, x \\ dx = \dfrac{dt}{1+t^2} \end{array}\right\} = \int \frac{1+t^2}{2+t} \cdot \frac{dt}{1+t^2} = \int \frac{dt}{t+2}$

$= \ln|t+2| = \ln|\mathrm{tg}\, x + 2|$; $\quad x \in \left(-\frac{\pi}{2}, -\mathrm{arctg}\, 2\right)$

$\left(-\mathrm{arctg}\, 2, \frac{\pi}{2}\right)$

a dále π-periodicky.

(45) $I_n = \int x^n e^{ax}$; $n \geq 0$ celé; $a \in \mathbb{R} \setminus \{0\}$.

$$I_{n+1} = \int \underbrace{x^{n+1}}_{u} \underbrace{e^{ax}}_{v'} dx = \frac{1}{a} x^{n+1} e^{ax} - (n+1) \int x^n \frac{1}{a} e^{ax} dx$$

$u' = (n+1) x^n$; $v = \frac{1}{a} e^{ax}$; $\boxed{I_{n+1} = \frac{x^{n+1} e^{ax}}{a} + \frac{n+1}{a} I_n}$

$$\boxed{I_0 = \int e^{ax} dx = \frac{1}{a} e^{ax}.}$$

v $\mathbb{R}$

(46) $I_n = \int \sin^n x \, dx$; $n \geq 0$ celé

$I_0 = \int dx = 1$; $I_1 = \int \sin x \, dx = -\cos x$

v $\mathbb{R}$

$$I_2 = \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) dx = \frac{x}{2} - \frac{1}{4} \sin 2x$$

rekurentní vzorec: (n ≥ 2 celé) $I_n = \int \underbrace{\sin x}_{u'} \cdot \underbrace{\sin^{n-1} x}_{v} dx$

$u = -\cos x$; $v' = (n-1) \sin^{n-2} x \cos x$

$$= -\cos x \sin^{n-1} x + (n-1) \int \underbrace{\cos^2 x}_{1 - \sin^2 x} \cdot \sin^{n-2} x \, dx$$

$$I_n = -\cos x \sin^{n-1} x + (n-1) \left[I_{n-2} - I_n \right]$$

$$\boxed{I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \left(\frac{n-1}{n} \right) I_{n-2} \quad \text{v } \mathbb{R}.}$$

(47) $J_n = \int \frac{dx}{\sin^n x} = \int \sin^{-n} x \, dx$; $n \geq 1$ celé

$$J_1 = \int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin\frac{x}{2} \cos\frac{x}{2}} = \int \frac{\cos\frac{x}{2}\, dx}{2 \sin\frac{x}{2} \cos^2\frac{x}{2}} \quad \Bigg| \begin{array}{l} y = \operatorname{tg}\frac{x}{2} \\ dy = \frac{dx}{2\cos^2\frac{x}{2}} \end{array}$$

$$= \int \frac{dy}{y} = \ln|y| = \ln\left|\operatorname{tg}\frac{x}{2}\right| ; \quad x \in (k\pi, (k+1)\pi).$$

$$J_2 = \int \frac{dx}{\sin^2 x} = -\operatorname{cotg} x \; ;$$

rekurentní vzorec (viz příklad (46) pro I_n výše).

$$J_n = I_{-n} \; ; \quad I_{-n} = \frac{-\cos x}{\sin^{n+1}} - (n+1) \cdot \left[I_{-(n+2)} - I_{-n} \right]$$

$$\boxed{J_{n+2} = \frac{-\cos x}{(n+2)\sin^{n+1} x} + \frac{n}{n+1} J_n}$$

(48) $I_n = \int \frac{dx}{(x^2+1)^n}$; $n \geq 1$ celé. $\quad I_1 = \operatorname{arctg} x$ v $\mathbb{R}$

$$I_n = \int \underbrace{1}_{u'} \cdot \underbrace{(x^2+1)^{-n}}_{v} dx = \frac{x}{(x^2+1)^n} + 2n \underbrace{\int \frac{x^2}{(x^2+1)^{n+1}} dx}_{I_n - I_{n+1}}$$

$u = x$; $v' = (-n) . 2x (x^2+1)^{-n-1}$

$$I_n = \frac{x}{(x^2+1)^n} + 2n\left[I_n - I_{n+1} \right]$$

$$\boxed{I_{n+1} = \frac{1}{2n} \cdot \frac{x}{(x^2+1)^n} + \frac{2n-1}{2n} I_n}$$