Risk Theory (NMFM503) – practicals

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1 Point processes – Poisson process

Basic notation a properties for the homogeneous Poisson process with intensity λ:

- N_t – random number of events observed until time t with distribution $\mathrm{Po}(\lambda\, t)$,
- σ_n – random time of $n-$th event with Erlang distribution with pdf

$$f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \; x \geq 0,$$

- τ_i – (independent) random time between events $i-1$ and i with exponential distribution $\mathrm{Exp}(\lambda)$.

Example 1.1. *Construct a random sample from Poisson distribution using the properties of the Poisson process.*
Hint. *Let* $\{U_k\}_{k=1,2,\dots}$ *be iid with uniform distribution on* $[0,1]$. *Set*

$$T_k := \prod_{i=1}^{k} U_i,$$

and

$$N := \inf\{n : T_n < e^{-\lambda}\}.$$

Show that $N - 1 \sim \mathrm{Po}(\lambda)$.

Example 1.2. *Let* n *events of homogenous Poisson process be observed during the time period* $[0,T]$ *at times* $\sigma_1 = s_1, \dots, \sigma_n = s_n$.

i. *Derive the maximum-likelihood estimate of the intensity* λ.

ii. *Verify the properties of the ML estimate: unbiasedness, consistency.*

iii. *Derive the sufficient statistic(s) for the estimate.*

iv. *Construct a confidence interval for the intensity.*

Example 1.3. *Let n events of homogenous Poisson process with parameter λ be observed during the time period $[0, T]$ at times $\sigma_1 = s_1, \ldots, \sigma_n = s_n$ and let n' events of homogenous Poisson process with parameter λ' be observed during the time period $[0, T']$ at times $\sigma'_1 = s'_1, \ldots, \sigma'_{n'} = s'_{n'}$. Let the processes be independent. Propose a statistical test of the hypothesis $\lambda = \lambda'$.*

Example 1.4. *Generalize the above test to k independent homogenous Poisson processes, i.e. derive a test of hypothesis $\lambda = \lambda_2 = \cdots = \lambda_k$.*

Example 1.5. *Let n events of* ***nonhomogenous*** *Poisson process be observed during the time period $[0, T]$ at times $\sigma_1 = s_1, \ldots, \sigma_n = s_n$. Consider intensity*

$$\lambda(t) = e^{\alpha + \beta\, t}.$$

Derive a statistical test of hypothesis $\beta = \beta_0$, and focus on the case $\beta_0 = 0$, i.e. construct a test of homogeneity.

2 Collective risk model and ruin probability, subexponential distributions

Example 2.1. *Under the standard assumptions (compound Poisson process, costs, premium, see the Lecture notes for details) derive the adjustment coefficient R when the severity distribution follows $\Gamma(\frac{1}{2}, \beta)$ with pdf*

$$p(x) = \frac{\sqrt{\beta}}{\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{-\beta x}.$$

Example 2.2. *Derive the Laplace transform of the Beekman's formula. Then compute the probability of ruin under the exponential distribution of claims severity, i.e.*

$$p(x) = a\, e^{-ax}.$$

Example 2.3. *Verify that the distribution with pdf*

$$p(x) = \frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{a^2}{2x}}$$

belongs to the subexponential family.

Example 2.4. *Show that the standard criterion does not show that the lognormal distribution belongs to the subexponential family.*

Example 2.5. (*) *Derive an asymptotic formula for the ruin probability under the lognormal distribution of the claim severity.*

Example 2.6. *Consider Excess of Loss (XL) reinsurance with priority $a > 0$ and layer $L > 0$. Let the claims of insurer follow the compound Poisson process. Elaborate the claims from the point of view of the reinsurer.*

Example 2.7. *Derive an estimate of the parameter of Pareto distribution based on the quantiles.*

3 Extreme Value Theory

Example 3.1. *Consider Fréchet distribution X with cdf*

$$G_{1,\alpha}(x) = \exp(-x^{-\alpha}),\ x > 0,\ \alpha > 0.$$

Verify that for the moments it holds

$$\mathbb{E}[X^j] = \Gamma(1 - \frac{j}{\alpha}),\ j < \alpha.$$

Example 3.2. *Verify the max-stability of the extreme value distributions, i.e. that for cdf G and proper choices of the sequences $\{c_n > 0\}$, $\{d_n\}$, it holds*

$$G^n(c_n x + d_n) = G(x),\ n = 1, 2, \ldots$$

Example 3.3. *Show that the extreme value distributions can be used to deal with the distributions of the minima of a sequence of random variables.*
Hint: Show that

$$P\left(\max_{i\le n}(-X_i) \le a_n x + b_n\right) = 1 - P\left(\min_{i\le n}(X_i) \le -a_n x - b_n\right),$$

where $a_n > 0$, $b_n \in \mathbb{R}$.

Example 3.4. *Consider generalized Pareto distribution with cdf*

$$W_{\gamma,\mu,\sigma}(x) = 1 - \left(1 + \gamma \frac{x-\mu}{\sigma}\right)^{-\frac{1}{\gamma}},$$

where $\gamma \neq 0$, $\mu \in \mathbb{R}$, $\sigma > 0$. *For* $\gamma > 0$ *the support is* $x \geq \mu$, *whereas for* $\gamma < 0$ *we have* $\mu \leq x \leq \mu - \frac{\sigma}{\gamma}$. *Show that*

$$\mathbb{E}[X] = \mu + \frac{\sigma}{1-\gamma}, \text{ if } \gamma < 1,$$

and

$$\mathbb{E}[X] = \infty, \text{ if } \gamma \geq 1.$$

Example 3.5. *Explore the limiting tail behaviour and derive the domain of attraction for the following distributions:*

1. *Pareto,*
2. *Exponential,*
3. *Beta.*

Example 3.6. *Consider a sequence of i.i.d. random variables with distribution function* $F(x)$. *Derive cdf for the maximum over random number of random variables, where the random number follows Poisson distribution with parameter* λ. *Then consider the case when* $F(x)$ *corresponds to generalized Pareto distribution with parameters* $(\gamma, 0, \sigma)$.

4 Copula functions

Example 4.1. *Consider bivariate discrete distribution with realization and probabilities*

$$P(X_1 = 0, X_2 = 0) = \frac{1}{8},\ P(X_1 = 1, X_2 = 1) = \frac{3}{8},$$
$$P(X_1 = 0, X_2 = 1) = \frac{2}{8},\ P(X_1 = 1, X_2 = 0) = \frac{2}{8}.$$

Derive the marginal distributions and discuss the (non)uniqueness of the copula function which represents the dependence.

Example 4.2. *Consider the multivariate and bivariate Gaussian copula. Derive an explicit formula for* $\rho \in \{-1, 0, 1\}$.

Example 4.3. *Consider the bivariate Gumbel copula with parameter* $\theta \in [1, \infty)$. *Compute the limit for* $\theta \to \infty$.

Example 4.4. *Consider the bivariate Clayton copula with parameter* $\theta \in (0, \infty)$. *Compute the limit for* $\theta \to \infty$ *and* $\theta \to 0_+$.