Optimization with application in finance – exercises

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HW 2021: Example 1.8 (any method), example 1.10 (KKT conditions)

3 Multiobjective optimization

We start with the notation of dominance for two-dimensional vectors.

Definition 3.1 *We say that* $\binom{x_1}{x_2}$ *strictly dominates* $\binom{y_1}{y_2}$, *denoted by* $\binom{x_1}{x_2} \succ \binom{y_1}{y_2}$, *iff* $x_1 < y_1$ *and* $x_2 < y_2$ [1]. *We say that* $\binom{x_1}{x_2}$ *dominates* $\binom{y_1}{y_2}$, *denoted by* $\binom{x_1}{x_2} \succeq \binom{y_1}{y_2}$, *iff* $x_1 \leq y_1$ *and* $x_2 \leq y_2$ *with at least one inequality strict. We say that* $\binom{x_1}{x_2}$ *is (weakly) efficient if there is no other* $\binom{y_1}{y_2}$ *such that* $\binom{y_1}{y_2} \succeq (\succ) \binom{x_1}{x_2}$.

Example 3.2 *Consider five pairs*

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ B = \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ C = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ D = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ E = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Identify all (weakly) efficient pairs.

Solution: We can identify pairs which are dominated, i.e.

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \succeq (\succ) \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \succeq (\succ) \begin{pmatrix} 3 \\ 3 \end{pmatrix},\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \succeq (\not\succ) \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Therefore, A, C, D are weakly efficient, whereas only A, D are efficient. □

We repeat the notion of optimality in multiobjective optimization.

Definition 3.3 *Consider multiobjective optimization problem*

$$\min_{x \in X} \left(f_1(x), \ldots, f_K(x) \right),$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$ *and* $X \subseteq \mathbb{R}^n$. *We say that* $\hat{x} \in X$ *is an efficient solution if there exists no other* $x \in X$ *such that* $f_k(x) \leq f_k(\hat{x})$ *for all* k *with at least one inequality strict. We denote by* $X^{eff} \subseteq X$ *the set of efficient solutions.*

Basic methods to find the efficient solutions are:

1. Aggregate function approach:

$$\min_{x \in X} \sum_{k=1}^{K} \lambda_k f_k(x),$$

   with parameters $\lambda_k \geq 0$.

[1] Since we will consider minimization problems later, lower values are preferred.

2. ε-constrained approach

$$\begin{aligned} &\min \; f_1(x) \\ &\text{s.t.} \; f_k(x) \leq \varepsilon_k, \; k = 2, \ldots, K, \\ &\qquad x \in X, \end{aligned}$$

where parameters ε_k are selected such that the problem is feasible.

3. Goal programming.

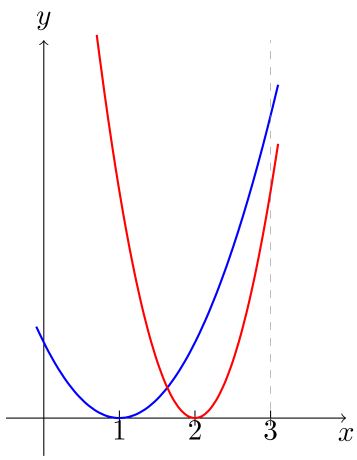
For basic properties see the lecture notes.

Example 3.4 *Consider biobjective nonlinear optimization problem*

$$\min_x \begin{pmatrix} (x-1)^2 \\ 3(x-2)^2 \end{pmatrix}$$
$$\text{s.t. } x \in [0,3].$$

Find all efficient solutions.

Solution: We can use the plot



By comparing the objective functions at different points of domain $[0, 3]$, we can identify the efficient solutions as interval $[1, 2]$. □

Example 3.5 *Consider biobjective nonlinear optimization problem*

$$\min_x \begin{pmatrix} 2(x+1)^2 \\ (x-2)^2 \end{pmatrix}$$
$$\text{s.t. } x \in [0,5].$$

Find all efficient solutions.

Example 3.6 *Consider biobjective linear optimization problem*

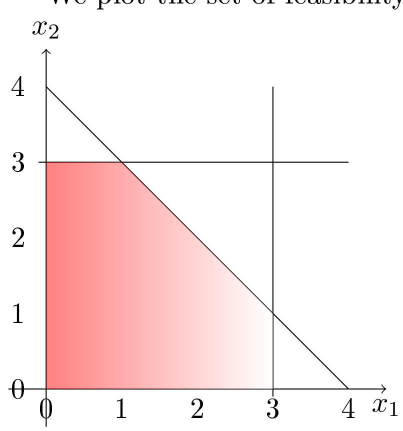
$$\begin{aligned} \min \; & \begin{pmatrix} -3x_1 - x_2 \\ x_1 - 2x_2 \end{pmatrix} \\ \text{s.t. } & x_1 + x_2 \leq 4, \\ & x_1 \leq 3, \\ & x_2 \leq 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Find all efficient solutions.

Solution: Denote by $f : \mathbb{R}^2 \to \mathbb{R}^2$ the vector objective, i.e.

$$f(x_1, x_2) = \begin{pmatrix} -3x_1 - x_2 \\ x_1 - 2x_2 \end{pmatrix}.$$

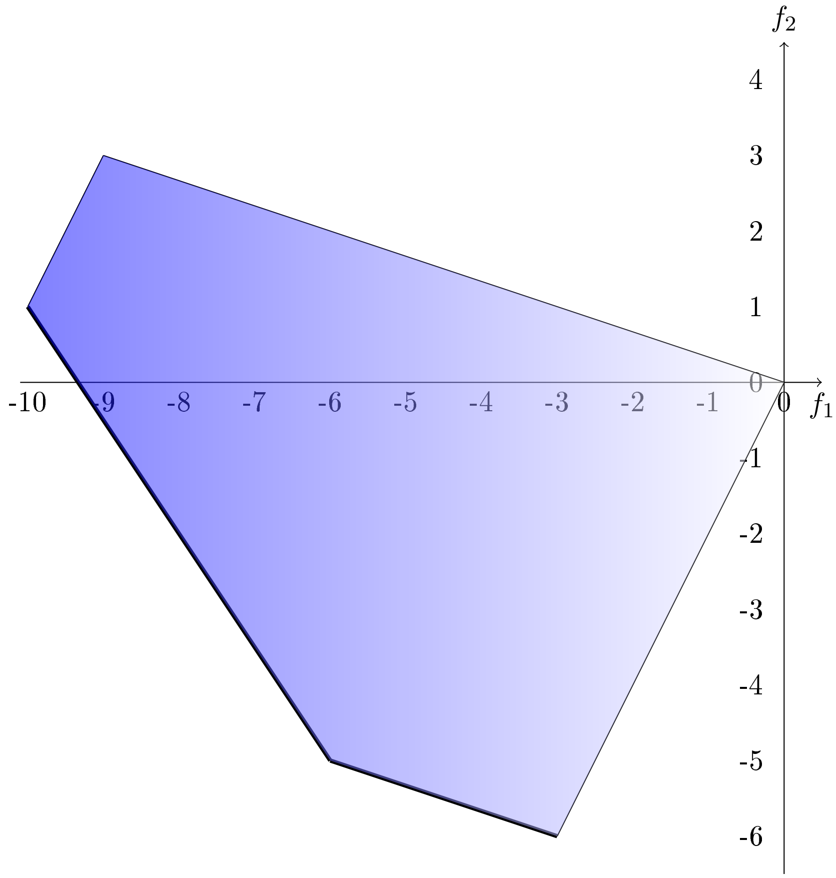
We plot the set of feasibility solutions



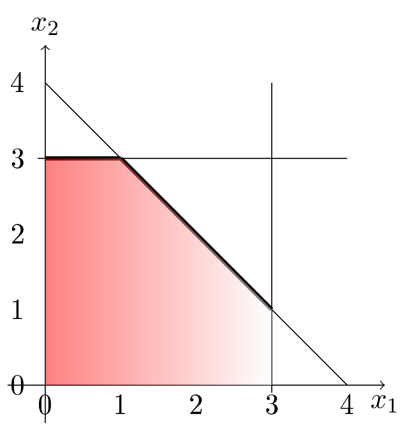
Using the picture of the feasibility set, we can identify the extreme points and compute their images, i.e.

$$f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; f(3,0) = \begin{pmatrix} -9 \\ 3 \end{pmatrix}, \; f(3,1) = \begin{pmatrix} -10 \\ 1 \end{pmatrix}, \; f(1,3) = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \; f(0,3) = \begin{pmatrix} -3 \\ -6 \end{pmatrix}.$$

These values can be then used to plot the image of the feasibility set.



Since we minimize the vector objective, we can identify the efficient frontier in the image space (in bold). Then we can return back to the decision vector space and identify the efficient solutions (in bold):



The set of efficient solutions can be written as

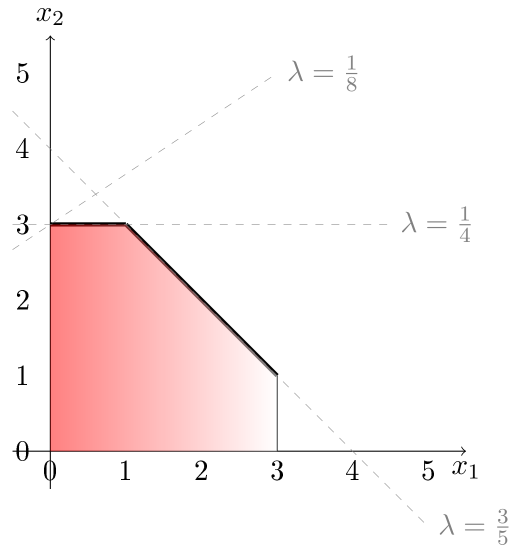
$$X^{eff} = \left\{ \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ \alpha \in [0,1] \right\} \cup \left\{ \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 0 \\ 3 \end{pmatrix},\ \alpha \in [0,1] \right\},$$

i.e. it is union of two edges. Realize that the set is not convex.

Instead of deriving the image of the feasibility set, we can use the aggregate function approach, i.e. minimize one objective

$$\lambda(-3x_1 - x_2) + (1-\lambda)(x_1 - 2x_2) = (-4\lambda + 1)x_1 + (\lambda - 2)x_2,$$

for different values of parameter $\lambda \in [0, 1]$.



We can observe from the picture that already values $\lambda = \frac{3}{5}$ and $\lambda = \frac{1}{4}$ identify the whole set of efficient solutions. □

Example 3.7 *Consider biobjective linear optimization problem*

$$\begin{aligned} \min &\begin{pmatrix} 3x_1 + x_2 \\ -x_1 - 2x_2 \end{pmatrix} \\ \text{s.t. } & x_2 \le 3, \\ & 3x_1 - x_2 \le 6, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Find all efficient solutions.

Solution: In this case, we are going to use the simplex algorithm, which is effective in the case of two (linear) objective functions. First, we will reformulate the problem in the standard form and apply the aggregation of objectives for $\lambda \in [0, 1]$:

$$\begin{aligned} \min\ & \lambda(3x_1 + x_2) + (1-\lambda)(-x_1 - 2x_2) = (4\lambda - 1)x_1 + (3\lambda - 2)x_2 \\ \text{s.t. } & x_2 + x_3 = 3, \\ & 3x_1 - x_2 + x_4 = 6, \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

We can start the simplex table with x_3, x_4 as the basic variables.

			$4\lambda-1$	$3\lambda-2$	0	0
			x_1	x_2	x_3	x_4
0	x_3	3	0	1	1	0
0	x_4	6	3	-1	0	1
		0	$-4\lambda+1$	$-3\lambda+2$	0	0
			$\leq 0 \Leftrightarrow \lambda \geq \frac{1}{4}$	$\leq 0 \Leftrightarrow \lambda \geq \frac{2}{3}$	≤ 0	≤ 0

The first table is already optimal for $\lambda \in [\frac{2}{3}, 1]$, i.e. $(0,0,3,6)$ is an optimal (efficient) solution. We can continue in iterations for values $\lambda \in [\frac{1}{4}, \frac{2}{3}]$. We change the basis: x_3 is replaced by x_2.

			$4\lambda-1$	$3\lambda-2$	0	0
			x_1	x_2	x_3	x_4
$3\lambda-2$	x_2	3	0	1	1	0
0	x_4	9	3	0	1	1
		$9\lambda-6$	$-4\lambda+1$	0	$3\lambda-2$	0
			$\leq 0 \Leftrightarrow \lambda \geq \frac{1}{4}$	≤ 0	$\leq 0 \Leftrightarrow \lambda \leq \frac{2}{3}$	≤ 0

The optimality condition is fulfilled for $\lambda \in [\frac{1}{4}, \frac{2}{3}]$, i.e. $(0,3,0,9)$ is an optimal (efficient) solution. We continue for $\lambda \in [0, \frac{1}{4}]$ by changing the basis: x_4 is replaced by x_1.

			$4\lambda-1$	$3\lambda-2$	0	0
			x_1	x_2	x_3	x_4
$3\lambda-2$	x_2	3	0	1	1	0
$4\lambda-1$	x_1	3	1	0	$\frac{1}{3}$	$\frac{1}{3}$
		$21\lambda-9$	0	0	$\frac{13\lambda-7}{3}$	$\frac{4\lambda-1}{3}$
			≤ 0	≤ 0	$\leq 0 \Leftrightarrow \lambda \leq \frac{7}{13}$	$\leq 0 \Leftrightarrow \lambda \leq \frac{1}{4}$

The optimality condition is fulfilled for $\lambda \in [0, \frac{1}{4}]$, i.e. $(3,3,0,0)$ is an optimal (efficient) solution. We went through all possible values of parameter λ.

We must be careful with values $\lambda \in \{0,1\}$, because then one objective function is not taken into account. Uniqueness of the optimal solution is then necessary to verify that it is an efficient solution. However, in our case the efficiency is ensured by that the values $\{0,1\}$ are contained in nontrivial intervals $\lambda \in [\frac{2}{3}, 1]$, and $\lambda \in [0, \frac{1}{4}]$, for which the solutions are stable.

If we return back to the original problem (by excluding slack variables x_3, x_4), we obtain the set of efficient solutions

$$X^{eff} = \left\{ \alpha \binom{0}{0} + (1-\alpha)\binom{0}{3},\ \alpha \in [0,1] \right\} \cup \left\{ \alpha \binom{0}{3} + (1-\alpha)\binom{3}{3},\ \alpha \in [0,1] \right\}.$$

Alternative way how to find the efficient solutions, is to use the ε-constrained approach when one objective is minimized and the other one is used as a constraint:

$$\begin{aligned} \min\ & 3x_1 + x_2 \\ \text{s.t.}\ & -x_1 - 2x_2 \le \varepsilon, \\ & x_2 \le 3, \\ & 3x_1 - x_2 \le 6, \\ & x_1, x_2 \ge 0, \end{aligned}$$

where ε is a parameter. This parameter must be restricted to the values when the problem is feasible. Realize that if the parametric problem is solved by the simplex algorithm, the parameter ε appears only in the column $B^{-1}b$. □

Example 3.8 *Consider biobjective linear optimization problem*

$$\begin{aligned} \min\ & \begin{pmatrix} 2x_1 - x_2 \\ x_1 - 2x_2 \end{pmatrix} \\ \text{s.t.}\ & -x_1 + x_2 \le 2, \\ & x_1 \le 2, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Find all efficient solutions.

Example 3.9 *Consider biobjective nonlinear optimization problem*

$$\begin{aligned} \min\ & \begin{pmatrix} -x_1 - 2x_2 \\ x_1^2 + 5x_2 - 1 \end{pmatrix} \\ \text{s.t.}\ & 4x_1 - x_2 \le 0, \\ & x_2 \le 8. \end{aligned}$$

Find all efficient solutions using the KKT optimality conditions.

Solution: We can use the aggregate function approach to transform the problem to a parametric optimization one, i.e. we solve for $\lambda \in [0, 1]$

$$\begin{aligned} \min\ & \lambda(-x_1 - 2x_2) + (1 - \lambda)(x_1^2 + 5x_2 - 1) \\ \text{s.t.}\ & 4x_1 - x_2 \le 0, \\ & x_2 \le 8. \end{aligned}$$

Note that the problem is convex. The Lagrange function is then

$$L(x_1, x_2, u_1, u_2) = \lambda(-x_1 - 2x_2) + (1 - \lambda)(x_1^2 + 5x_2 - 1) + u_1(4x_1 - x_2) + u_2(x_2 - 8),$$

with $u_{1,2} \geq 0$. The KKT optimality conditions are

$$\begin{aligned} i)\ & 4x_1 - x_2 \leq 0,\ \ x_2 - 8 \leq 0, \\ ii)\ & u_1(4x_1 - x_2) = 0,\ \ u_1 \geq 0, \\ & u_2(x_2 - 8) = 0,\ \ u_2 \geq 0, \\ iii)\ & \frac{\partial L}{\partial x_1} = -\lambda + 2(1-\lambda)x_1 + 4u_1 = 0, \\ & \frac{\partial L}{\partial x_2} = -2\lambda + 5(1-\lambda) - u_1 + u_2 = 0. \end{aligned}$$

We can split our solution to four cases according to the complementarity conditions:

1) $u_1 = 0$, $u_2 = 0$: from iii) we have $5 - 7\lambda = 0$, i.e. parameter is restricted to $\lambda = \frac{5}{7} \in [0,1]$. From iii) we also obtain

$$x_1 = \frac{\lambda}{2(1-\lambda)} = \frac{5}{4}.$$

To get a feasible solution, we must restrict x_2 using i), i.e.

$$4x_1 - x_2 \leq 0 \Leftrightarrow x_2 \geq 5 \text{ and also } x_2 \leq 8.$$

Thus we have KKT point

$$\left(\frac{5}{4},\ x_2 \in [5,8],\ 0,\ 0\right).$$

2) $u_1 = 0$, $x_2 = 8$: From iii) we have

$$x_1 = \frac{\lambda}{2(1-\lambda)},$$

and

$$u_2 = 7\lambda - 5 \geq 0 \Leftrightarrow \lambda \geq \frac{5}{7}.$$

From i) it must hold $4x_1 \leq 8$, i.e.

$$x_1 = \frac{\lambda}{2(1-\lambda)} \leq 2 \Leftrightarrow \lambda \leq \frac{4}{5}.$$

Thus we have KKT point

$$\left(\frac{\lambda}{2(1-\lambda)},\ 8,\ 0,\ 7\lambda - 5\right),\ \lambda \in \left[\frac{5}{7}, \frac{4}{5}\right].$$

3) $4x_1 - x_2 = 0$, $u_2 = 0$: From iii) we have

$$u_1 = -7\lambda + 5 \geq 0 \Leftrightarrow \lambda \leq \frac{5}{7}.$$

Using iii)

$$-\lambda + 2(1-\lambda)x_1 + 4(-7\lambda + 5) = 0 \Leftrightarrow x_1 = \frac{29\lambda - 20}{2(1-\lambda)}.$$

From ii) $x_2 = 4x_1$ and from i)

$$x_2 \le 8 \Leftrightarrow \lambda \le \frac{8}{11} > \frac{5}{7}.$$

Thus we have KKT point

$$\left(\frac{29\lambda - 20}{2(1-\lambda)},\ \frac{58\lambda - 40}{1-\lambda},\ -7\lambda + 5,\ 0\right),\ \lambda \in \left[0, \frac{5}{7}\right].$$

4) $4x_1 - x_2 = 0$, $x_2 = 8$, i.e. $x_1 = 2$: From iii) we have

$$\begin{aligned} &-\lambda + 4(1-\lambda) + 4u_1 = 0,\\ &-2\lambda + 5(1-\lambda) - u_1 + u_2 = 0. \end{aligned}$$

We multiply the second equation by 4 and sum the equations, we derive

$$u_2 = \frac{33\lambda - 24}{4} \ge 0 \Leftrightarrow \lambda \ge \frac{8}{11},$$

and

$$u_1 = \frac{5\lambda - 4}{4} \ge 0 \Leftrightarrow \lambda \ge \frac{4}{5} > \frac{8}{11}.$$

Thus we have KKT point

$$\left(2,\ 8,\ \frac{5\lambda - 4}{4},\ \frac{33\lambda - 24}{4}\right),\ \lambda \in \left[\frac{4}{5}, 1\right].$$

Each KKT point represents an efficient solution of the multiobjective problem (for $\lambda = 0$ after discussion). □

Example 3.10 *Consider biobjective nonlinear optimization problem*

$$\begin{aligned} \min &\begin{pmatrix} (x_1 - 2)^2 + x_2^2 \\ x_1^2 + (x_2 - 2)^2 \end{pmatrix} \\ \text{s.t. } & x_1 + x_2 \le 1. \end{aligned}$$

Find all efficient solutions using the KKT optimality conditions.