Optimization with application in finance – exercises

Martin Branda, 21 February 2023

1 Parametric linear optimization

1.1 Simplex algorithm

Example 1.1 *Consider linear programming problem*

$$\begin{aligned} \min\ & 2x_1 - x_2 \\ \text{s.t.}\ & -x_1 + x_2 \le 1, \\ & x_2 \le 3, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Solve the problem using the simplex algorithm.

Solution:

			2	−1	0	0
			x_1	x_2	x_3	x_4
0	x_3	1	−1	1	1	0
0	x_4	3	0	1	0	1
		0	−2	1	0	0
−1	x_2	1	−1	1	1	0
0	x_4	2	1	0	−1	1
		−1	−1	0	−1	0

The optimal solution is $(0,1,0,2)$ with optimal value -1.

1.2 Postoptimization

Example 1.2 *Consider linear programming problem*

$$\begin{aligned} \min\ & 2x_1 - x_2 \\ \text{s.t.}\ & -x_1 + x_2 \le 1, \\ & x_2 \le 3, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Solve the problem using the simplex algorithm. Then investigate the stability with respect to

1. *objective function:* $c = (-1, -1)$,
2. *new decision variable* x_5*:* $c_5 = -2$, $a_{\bullet 5} = (1, \frac{3}{2})$,

3.* *right hand side vector:* $b = (1, 0.5)$,

4.* *new constraint:* $x_2 \le \frac{1}{2}$.

1.3 Parametric linear programming

Example 1.3 *Consider the linear programming problem with real parameter* λ

$$\begin{aligned} &\min\ 3x_1 + 5x_2 \\ &\text{s.t. } 2x_1 + x_2 \geq 10, \\ &\quad x_1 + 2x_2 \geq 12 + \lambda, \\ &\quad x_1 + x_2 \geq 8, \\ &\quad x_{1,2} \geq 0. \end{aligned}$$

Using the graphical method, find the optimal solution and optimal values in dependence on the values of λ*.*

Example 1.4 *Consider the linear programming problem with real parameter* λ

$$\begin{aligned} &\min\ 2x_1 - x_2 \\ &\text{s.t. } -x_1 + \lambda x_2 + x_3 = 1, \\ &\quad x_2 + x_4 = 3, \\ &\quad x_{1,2,3,4} \geq 0. \end{aligned}$$

Discuss an iteration of suitable simplex algorithm in dependence on the values of λ*.*

Solution: We can start with the simplex table

			2	-1	0	0
			x_1	x_2	x_3	x_4
0	x_3	1	-1	λ	1	0
0	x_4	3	0	1	0	1
		0	-2	1	0	0

If $\lambda \geq \frac{1}{3}$, x_2 replaces x_3 in the basis, whereas if $\lambda < \frac{1}{3}$, x_2 replaces x_4 in the basis. In the first case, we get

			2	-1	0	0
			x_1	x_2	x_3	x_4
-1	x_2	$\frac{1}{\lambda}$	$\frac{-1}{\lambda}$	1	$\frac{1}{\lambda}$	0
0	x_4	$3 - \frac{1}{\lambda}$	$\frac{1}{\lambda}$	0	$\frac{-1}{\lambda}$	1
		$\frac{-1}{\lambda}$	$\frac{1}{\lambda} - 2$	1	$\frac{-1}{\lambda}$	0

If $\lambda \geq \frac{1}{2}$, then the optimality condition is fulfilled and we have got an optimal solution. When $\lambda \in [\frac{1}{3}, \frac{1}{2})$, then we continue with iterations and x_1 replaces x_4 in the basis.

Dual simplex algorithm*

Primal problem (standard form)

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b, \\ & x \geq 0. \end{aligned}$$

Basis B = regular square submatrix of A, i.e. A can be divided into the basis and nonbasis part

$$A = (B|N).$$

We also consider $B = \{i_1, \dots, i_m\}$ as the set of column indices which correspond to the basis. We split also the objective coefficients and the decision vector accordingly:

$$c^T = (c_B^T, c_N^T),$$
$$x^T(B) = (x_B^T(B), x_N^T(B)),$$

where

$$x_B(B) = B^{-1}b,\ x_N(B) \equiv 0.$$

We consider

- feasible basis for which $x_B(B) \geq 0$ (and $x_N(B) = 0$),
- optimal basis corresponding to an optimal solution,
- basic solution(s).

The simplex algorithm can be represented by the simplex table:

			x^T
			c^T
c_B	$x_B(B)$	$B^{-1}b$	$B^{-1}A$
		$c_B^T B^{-1} b$	$c_B^T B^{-1} A - c^T$

In the table, we can identify

- feasibility condition:
$$B^{-1}b \geq 0,$$
- optimality condition:
$$c_B^T B^{-1} A - c^T \leq 0.$$

Dual problem

$$\begin{aligned} &\max\ b^T y \\ &\text{s.t. } A^T y \leq c, \\ &\quad y \in \mathbb{R}^m. \end{aligned}$$

Dual simplex algorithm works with **dual feasible basis** B and **basic dual solution** $y(B)$, forwhich it holds

$$\begin{aligned} B^T y(B) &= c_B, \\ N^T y(B) &\leq c_N. \end{aligned}$$

Primal feasibility $B^{-1}b \geq 0$ **is violated** until reaching the optimal solution. **Primal optimality condition = dual feasibility** is always fulfilled:

$$c_B^T B^{-1} A - c^T \leq 0.$$

Using notation $A = (B|N)$, $c^T = (c_B^T, c_N^T)$, we have

$$\begin{aligned} c_B^T B^{-1} B - c_B^T &= 0, \\ c_B^T B^{-1} N - c_N^T &\leq 0, \end{aligned}$$

Setting $\hat{y} = (B^{-1})^T c_B$

$$\begin{aligned} B^T \hat{y} &= c_B^T, \\ N^T \hat{y} &\leq c_N^T. \end{aligned}$$

Thus, $\hat{y}$ is a basic dual solution.

Dual simplex algorithm – a step:

- Find index $u \in B$ such that $x_u(B) < 0$ and denote the corresponding row by

$$\tau^T = (B^{-1}A)_{u,\bullet}.$$

- Denote the criterion row by

$$\delta^T = c_B^T B^{-1} A - c^T \leq 0.$$

- Minimize the ratios

$$\hat{i} = \arg\min \left\{ \frac{\delta_i}{\tau_i} :\ \tau_i < 0 \right\}.$$

- Substitute x_u by $x_{\hat{i}}$ in the basic variables, i.e. $\hat{B} = B \setminus \{u\} \cup \{\hat{i}\}$. We move to another **basic dual solution**.

We say that the problem is **dual nondegenerate** if for all dual feasible basis B it holds

$$(A^T y(B) - c)_j = 0, \; j \in B,$$
$$(A^T y(B) - c)_j < 0, \; j \notin B.$$

If the problem is dual nondegenerate, then the dual simplex algorithm ends after finitely many steps.

Example 1.5 *Using the dual simplex algorithm solve the following linear programming problem*

$$\begin{aligned} \min x_1 + x_2 & \\ 2x_1 + x_2 &\geq \frac{3}{2}, \\ x_1 + x_2 &\geq 1, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution: We will solve the problem in the following standard form

$$\begin{aligned} &\min \; x_1 + x_2 \\ &\quad -2x_1 - x_2 + x_3 = -\frac{3}{2}, \\ &\quad -x_1 - x_2 + x_4 = -1, \\ &\quad x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

We can derive the dual problem

$$\begin{aligned} \max \; & -\frac{3}{2}y_1 - y_2 \\ \text{s.t.} \; & -2y_1 - y_2 \leq 1 \\ & -y_1 - y_2 \leq 1 \\ & y_1 \leq 0 \\ & y_2 \leq 0. \end{aligned}$$

			1	1	0	0
			x_1	x_2	x_3	x_4
0	x_3	$-\frac{3}{2}$	-2	-1	1	0
0	x_4	-1	-1	-1	0	1
		0	-1	-1	0	0
1	x_1	$\frac{3}{4}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0
0	x_4	$-\frac{1}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1
		$\frac{3}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
1	x_1	$\frac{1}{2}$	1	0	-1	1
1	x_2	$\frac{1}{2}$	0	1	1	-2
		1	0	0	0	-1

In the final table, we can identify the optimal solutions of

- primal problem: $(\frac{1}{2}, \frac{1}{2}, 0, 0)$,
- dual problem: $(0, -1)$.

Optimal value is equal to 1.

Example 1.6 *Using the dual simplex algorithm solve the following linear programming problem*

$$\begin{aligned} \min 4x_1 + 5x_2 & \\ x_1 + 4x_2 &\geq 5, \\ 3x_1 + 2x_2 &\geq 7, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution: We can formulate the dual problem

$$\begin{aligned} \max\ & -5y_1 - 7y_2 \\ \text{s.t.}\ & -y_1 - 3y_2 \leq 4 \\ & -4y_1 - 2y_2 \leq 5 \\ & y_1 \leq 0 \\ & y_2 \leq 0. \end{aligned}$$

			4	5	0	0
			x_1	x_2	x_3	x_4
0	x_3	-5	-1	-4	1	0
0	x_4	-7	-3	-2	0	1
		0	-4	-5	0	0
0	x_3	-8/3	0	-10/3	1	-1/3
4	x_1	7/3	1	2/3	0	-1/3
		28/3	0	-7/3	0	-4/3
5	x_2	8/10	0	1	-3/10	1/10
4	x_1	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

The last solution is primal and dual feasible, thus optimal, i.e. $(18/10, 8/10)$ is the optimal solution of (P).

Example 1.7 *(*) Consider the linear programming problem with real parameter* λ

$$\begin{aligned} \min\ & 2x_1 - x_2 + x_3 \\ \text{s.t.}\ & x_1 + \lambda x_2 + x_3 = 2, \\ & x_1 - (2+\lambda)x_2 + x_4 = -1, \\ & x_{1,2,3,4} \geq 0. \end{aligned}$$

Discuss an iteration of suitable simplex algorithm in dependence on the values of λ*.*

Solution: We can start with the simplex table

			2	-1	1	0
			x_1	x_2	x_3	x_4
1	x_3	2	1	λ	1	0
0	x_4	-1	1	$-2-\lambda$	0	1
		2	-1	$\lambda+1$	0	0

We can observe that if $\lambda \leq -1$ then the optimality (=dual feasibility) is fulfilled, however the primal feasibility do not hold. There is only one possible pivot element $-2-\lambda$ which is negative only if $\lambda > -2$. So, if $\lambda \in (-2, -1]$, we can continue with iterations using the dual simplex algorithm. Basic variable x_4 is removed from the basis and x_2 enters

			2	-1	1	0
			x_1	x_2	x_3	x_4
1	x_3	2	1	λ	1	0
0	x_4	-1	1	$-2-\lambda$	0	1
		2	-1	$\lambda+1$	0	0
1	x_3	$\frac{\lambda+4}{\lambda+2}$	$\frac{2\lambda+2}{\lambda+2}$	0	1	$\frac{\lambda}{\lambda+2}$
-1	x_2	$\frac{1}{\lambda+2}$	$\frac{-1}{\lambda+2}$	1	0	$-\frac{1}{\lambda+2}$
		$\frac{\lambda+3}{\lambda+2}$	$-\frac{1}{\lambda+2}$	0	0	$\frac{\lambda+1}{\lambda+2}$

Remind that $\lambda \in (-2, -1]$. Since the criterion row is nonpositive, the primal optimality (= dual feasibility) is preserved. Moreover, the primal feasibility (= dual optimality) is fulfilled.

Example 1.8 *Consider the simplex table with real parameter* λ

			3	*-1*	*0*	*0*
			x_1	x_2	x_3	x_4
-1	x_2	$2-\lambda$	-1	*1*	*1*	*0*
0	x_4	*3*	*1*	*0*	-1	*1*
		$\lambda-2$	-2	*0*	-1	*0*

Discuss optimality in dependence on the values of λ *and perform one additional iteration of suitable simplex algorithm.*