Optimal outflow boundary condition for a stationary flow of an incompressible fluid

Michal Bathory and Miroslav Bulíček

Abstract. We prove the existence of a weak solution to the stationary Navier-Stokes system with an implicitly prescribed outflow boundary condition. This boundary condition is stated as a requirement that the "right" weak solution is a minimum of an appropriate functional. This functional can be chosen as, e.g., the energy dissipation, which then gives a physical meaning to such an implicit boundary condition. The existence theorem is proven in a general framework of non-Newtonian fluids. Further, we obtain an explicit form of the implicit boundary condition for two simpler fluid models: Stokes model with a non-linearity obeying the square integrable structure, and the Stokes model with the power-law rheology. For these models, the uniqueness of the weak solution with explicit or implicit outflow boundary condition is shown. This work builds upon and extends the idea presented in author's previous short article in proceedings, where only the Stokes model was considered.

1. Introduction

There is an old and very well known problem in fluid mechanics: How to prescribe an outflow boundary condition? This problem is definitely interesting from the viewpoint of engineering, numerical mathematics, physics and also mathematical analysis. Although its notoriety, there are many unanswered questions even in the simple setting of the stationary flow of an incompressible Newtonian fluid. The difficulty is, of course, that no apriori information about the outflow is available. Let us imagine the canonical situation of a flow through an elongated pipe described by, let us say, the Navier-Stokes system. Then, typically, one would like to prescribe an outflow boundary condition which ensures:

(M) existence (and uniqueness) of the solution,
(P) the boundary condition itself has a meaningful physical interpretation,
(E) admissibility of some reference flow (e.g. the Poiseuille flow),
(N) no matter, where the pipe is cut, the flow remains (almost) the same.

2010 *Mathematics Subject Classification.* 76D03; 76A02.

Key words and phrases. fluid mechanics, incompressible flow, outflow boundary condition, do-nothing boundary condition.

Supported by the Charles University, project GA UK No. 584217 and by the Nečas center for mathematical modelling.

It seems that there are certain general relations between these properties. For example, if (P) is satisfied, then one should be able to obtain an energy estimate, which is crucial for (M). Further, the properties (E) and (N) can be enforced by certain artificial alterations to the outflow boundary condition. This freedom of choice is probably related to the fact that properties (E) and (N) depend on how the geometry of the pipe continues behind the outlet (while (M) and (P) not). What is the real problem is although one can find many boundary conditions satisfying (M), (E), (N), the physical interpretation is lacking. Let us now substantiate this heuristics in the context of some outflow boundary conditions that are known.

The outflow boundary condition that one can find in the literature very often is the "do-nothing" boundary condition, which reads as

$$-p\boldsymbol{n}+\nu(\nabla\boldsymbol{v})\boldsymbol{n}=c\boldsymbol{n} \quad \text{on} \quad \Theta, \tag{1.1}$$

where $(\boldsymbol{v},p)$ represent the velocity and the pressure of the flow, $\nu>0$ stands for the kinematic viscosity, $\boldsymbol{n}$ denotes the outward normal vector on the outlet boundary Θ and where $c\in\mathbb{R}$ is some constant. Along with this condition, it is common to find a statement saying that "do-nothing" is a natural outflow boundary condition as it can be read from the weak formulation of the Navier-Stokes equation. However, it would be fair to say that there are infinitely many boundary conditions with this property. Indeed, since the incompressibility constraint $\operatorname{div}\boldsymbol{v}=0$ is assumed, we get $\operatorname{div}(\nabla\boldsymbol{v})^T=0$, which, in turn implies, for example, that

$$\begin{aligned}\int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{\varphi} &= \int_\Omega(-\nabla p+\nu\Delta\boldsymbol{v})\cdot\boldsymbol{\varphi}=\int_\Omega \operatorname{div}\boldsymbol{T}\cdot\boldsymbol{\varphi}\\ &=\int_\Theta(-p\mathbf{I}+2\nu\mathbf{D}\boldsymbol{v})\boldsymbol{n}\cdot\boldsymbol{\varphi}-2\nu\int_\Omega\mathbf{D}\boldsymbol{v}\cdot\mathbf{D}\boldsymbol{\varphi}\end{aligned}$$

for all $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,2}_{\Gamma,\operatorname{div}}(\Omega)$, where $\mathbf{D}\boldsymbol{v}:=\frac{1}{2}(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T)$ and $\boldsymbol{T}:=-p\mathbf{I}+2\nu\mathbf{D}\boldsymbol{v}$. According to this computation, we could prescribe

$$\boldsymbol{T}\boldsymbol{n}=c\boldsymbol{n} \quad \text{on} \quad \Theta \tag{1.2}$$

instead of (1.1). Then, the actual effect of prescribing (1.2) instead of (1.1) in the weak formulation of the Navier-Stokes equation is that the bilinear form $(\boldsymbol{v},\boldsymbol{\varphi})\mapsto\nu\int_\Omega\nabla\boldsymbol{v}\cdot\boldsymbol{\varphi}$ changes to $(\boldsymbol{v},\boldsymbol{\varphi})\mapsto 2\nu\int_\Omega\mathbf{D}\boldsymbol{v}\cdot\mathbf{D}\boldsymbol{\varphi}$. The reason why (1.1) is usually preferred over (1.2) (and over all other boundary condition obtained in this manner) is that it has the property (E), while, for example, (1.2) does not (see, e.g., [**8**] for details and corresponding figures). However, the great downside is that there is absolutely no physical reason for (1.1), i.e., it does not fulfil (P). Furthermore, neither of the conditions (1.1) and (1.2) satisfies (M) (there is no large-data existence result for Navier-Stokes system with these conditions). The cause for this lies in the fact that the corresponding energy estimate is not available.

There has been a lot of effort to find outflow boundary condition, for which the energy estimate holds. This usually results in a modification of (1.1), where the head pressure $p+\frac{1}{2}|\boldsymbol{v}|^2$ (cf. [**8**]) or a back flow $(\boldsymbol{v}\cdot\boldsymbol{n})_-$ (see [**4**] and [**3**]) is used. In [**4**], the authors find a whole family of energy-preserving boundary conditions. For illustration, one of them reads as

$$(\boldsymbol{T}-\boldsymbol{T}_P)\boldsymbol{n}+\tfrac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{n})_-(\boldsymbol{v}-\boldsymbol{v}_P)=0 \quad \text{on} \quad \Theta, \tag{1.3}$$

where quantities with subscript P correspond to the reference Poiseuille flow (in a straight pipe). This way, the property (E) is obviously satisfied. Furthermore, it is shown in [**4**] that also (M) and (N) are fulfilled. However, it is totally unclear whether (1.3) can be physically justified (its derivation in [**4**] is purely mathematical).

We shall, on the other hand present an (implicit) outflow boundary condition which can be physically interpreted and which ensures the existence of a weak solution. However, we will not be concerned with properties (E), (N), since these depend too much on a particular situation (model, geometry of the domain, Reynolds number, etc.)

The paper is organized as follows. The second section contains all the necessary notation and function spaces that are used. We also include some auxiliary technical results. In the third section, we shall introduce an (implicit) outflow boundary condition that fulfils (P). The fifth section contains the existence proof for quite a general fluid model with this implicit boundary condition. In the final section, we find an explicit form of this boundary condition for two types of the generalized Stokes model.

2. Mathematical tools

In this section, we shall fix the notation and review some basic concepts that are used in connection with the weak solutions to Navier-Stokes equations. Also, we include some generalizations of the convergence lemma that will be used to identify some weak non-linear limits.

2.1. Notation. The symbol Ω will always stand for an open, connected subset of $\mathbb{R}^d$, $d \geqslant 2$, with a Lipschitz boundary $\partial\Omega$. Then, the symbol $\boldsymbol{n}$ denotes the outward unit normal vector on $\partial\Omega$ (if it exists). By $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, we denote the set of all symmetric $n \times n$ matrices. We reserve the symbols $\boldsymbol{A}$ and $\boldsymbol{B}$ for symmetric matrices that are arbitrary, i.e., whenever a statement contains $\boldsymbol{A}$ or $\boldsymbol{B}$, it is meant for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Non-scalar quantities will be distinguished by a boldface. The inequality $L \leqslant cR$ will be sometimes shortened to $L \lesssim R$ if the number $c > 0$ is unimportant (i.e., if it is independent of quantities that we want to estimate). The weak and strong convergence will be distinguished just by $\rightharpoonup$ and $\to$, respectively.

2.2. Function spaces. For $1 \leqslant r \leqslant \infty$, we set $r' = \frac{r}{r-1}$. The Lebesgue space $L^r(\Omega)$, the Sobolev space $W^{1,r}(\Omega)$ and their vectorial counterparts $\boldsymbol{L}^r(\Omega)$, $\boldsymbol{W}^{1,r}(\Omega)$ are defined in an usual way. The set of all smooth functions of compact support in Ω is denoted by $C^\infty(\Omega)$ or $\boldsymbol{C}^\infty(\Omega)$. By the symbols $\boldsymbol{W}^{1,r}_0(\Omega)$ and $\boldsymbol{W}^{1,r}_{\mathrm{div}}(\Omega)$, we denote the subspaces of $\boldsymbol{W}^{1,r}(\Omega)$ of functions with zero trace and zero divergence, respectively. Further, the subset of $\boldsymbol{W}^{1,r}(\Omega)$ consisting of divergence-free functions whose trace is zero on some measurable subset Γ of the boundary $\partial\Omega$ is denoted by

$\boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$. We shall also need the fractional Sobolev space $\boldsymbol{W}^{1-\frac{1}{r},r}(\partial\Omega)$ for functions defined on the boundary $\partial\Omega$ (traces of $\boldsymbol{W}^{1,r}(\Omega)$ functions). The space of traces of functions from $\boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$ will be denoted correspondingly by $\boldsymbol{W}^{1-\frac{1}{r},r}_{\Gamma,\mathrm{div}}(\partial\Omega)$. It is the subspace of $\boldsymbol{W}^{1-\frac{1}{r},r}(\partial\Omega)$, whose elements $\boldsymbol{v}$ satisfy $\boldsymbol{v} = 0$ on Γ and $\int_{\partial\Omega} \boldsymbol{v}\cdot\boldsymbol{n} = 0$ (see Lemma 5.1 for details). The norms in $\boldsymbol{L}^r(\Omega)$ and $\boldsymbol{W}^{1,r}(\Omega)$ (and in their subspaces) will be denoted by $\|\cdot\|_r$ and $\|\cdot\|_{1,r}$, respectively. For any Banach space X, the symbol X^* stands for its topological dual space.

2.3. Maximally monotone graphs. Let $r \geqslant 1$. We shall say that $\boldsymbol{S} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ defines a maximal monotone r-graph if it fulfils each of the following properties:

$$(\boldsymbol{S}(\boldsymbol{A}) - \boldsymbol{S}(\boldsymbol{B}))\cdot(\boldsymbol{A}-\boldsymbol{B}) \geqslant 0$$

$$\text{if}\quad (\boldsymbol{S}(\boldsymbol{A}) - \overline{\boldsymbol{S}})\cdot(\boldsymbol{A}-\boldsymbol{B}) \geqslant 0, \quad \text{then}\quad \overline{\boldsymbol{S}} = \boldsymbol{S}(\boldsymbol{B})$$

$$c_1(|\boldsymbol{A}|^r + |\boldsymbol{S}(\boldsymbol{A})|^{r'}) - c_2 \leqslant \boldsymbol{S}(\boldsymbol{A})\cdot\boldsymbol{A}, \quad c_1, c_2 > 0.$$

2.4. Generalizations of the convergence lemma for a sequence of graphs. Suppose that we manage to describe the problem of interest using a maximally monotone graph and also suppose that we approximated the solution to this problem by some weakly converging sequence in this graph. Then, since the problem is non-linear, one needs to identify the weak limit, i.e., to show that the graph is closed in the weak topology that we considered. To do that, it seems natural to use the convergence lemma (cf. [**6**]) that tells us that it is enough to verify

$$\limsup_{\lambda\to\infty} \int_\Omega \boldsymbol{S}(\boldsymbol{D}_\lambda)\cdot\boldsymbol{D}_\lambda \leqslant \int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{D},$$

where $\overline{\boldsymbol{S}}$ and $\boldsymbol{D}$ denote the weak limits of $\boldsymbol{S}(\boldsymbol{D}_\lambda)$ and $\boldsymbol{D}_\lambda$, respectively.

However, we shall need a modified version of the convergence lemma for the case, where the graph itself changes with the index of the sequence. The next lemma will be used to identify the non-linear weak limit for the sequences of graphs that linearize the original graph near the origin.

Lemma 2.1. *Let $\{\boldsymbol{S}_\delta\}_{\delta>0}$ be a sequence of maximal monotone r-graphs, $r > 1$, satisfying*

$$\boldsymbol{S}_\delta(\boldsymbol{A})\cdot\boldsymbol{A} \geqslant c_1(|\boldsymbol{A}|^r + |\boldsymbol{S}_\delta(\boldsymbol{A})|^{r'}) - c_2,$$

where $c_1, c_2 > 0$ are independent of δ. Suppose that the graphs $\boldsymbol{S}_\delta$ converge to some graph $\boldsymbol{S}$ in the sense that

$$\boldsymbol{S}_\delta(\boldsymbol{A}) \to \boldsymbol{S}(\boldsymbol{A}) \quad \forall \boldsymbol{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{2.1}$$

If

$$\boldsymbol{D}_\delta \rightharpoonup \boldsymbol{D} \quad \text{in} \quad \boldsymbol{L}^r(\Omega),$$

$$\boldsymbol{S}_\delta(\boldsymbol{D}_\delta) \rightharpoonup \overline{\boldsymbol{S}} \quad \text{in} \quad \boldsymbol{L}^{r'}(\Omega).$$

and

$$\limsup_{\delta\to0+} \int_\Omega \boldsymbol{S}_\delta(\boldsymbol{D}_\delta)\cdot\boldsymbol{D}_\delta \leqslant \int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{D},$$

then

$$\overline{\boldsymbol{S}} = \boldsymbol{S}(\boldsymbol{D}).$$

PROOF. The assumption (2.1) implies

$$\boldsymbol{S}_\delta(\boldsymbol{D}) \to \boldsymbol{S}(\boldsymbol{D}) \quad \text{in} \quad \boldsymbol{L}^{r'}(\Omega). \tag{2.2}$$

Then

$$0 \leqslant \int_\Omega (\boldsymbol{S}_\delta(\boldsymbol{D}_\delta) - \boldsymbol{S}_\delta(\boldsymbol{D})) \cdot (\boldsymbol{D}_\delta - \boldsymbol{D})$$

and, using (2.2), one can repeat the proof of the usual convergence lemma (cf. proof of [**6**, Lemma 2.4.1]). □

In the next lemma we treat the other case, where the graph is linearized for large arguments. In this case, to preserve monotonicity of the graph, the coercivity estimate may contain some power-weights, which, however, cannot be arbitrary.

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\boldsymbol{S}$ be a maximally monotone r-graph with $r \in (1, \infty)$. Furthermore, let S_λ, $\lambda > 0$, be a graph defined by*

$$\mathsf{S}_\lambda(\boldsymbol{A}) := \begin{cases} \boldsymbol{S}(\boldsymbol{A}), & |\boldsymbol{A}| \leqslant \lambda; \\ \boldsymbol{N}_\lambda(\boldsymbol{A}), & |\boldsymbol{A}| > \lambda, \end{cases}$$

where $\boldsymbol{N}_\lambda$ is a monotone 2-graph with $\boldsymbol{N}_\lambda(0) = 0$ and chosen in a way that S_λ is a monotone 2-graph, for each $\lambda > 0$.

Let

$$\alpha > -\frac{1}{2}$$

and set

$$\beta := 2\alpha + 2.$$

Let $\{\boldsymbol{D}_\lambda\}_{\lambda>0}$ be a sequence in $\boldsymbol{L}^1(\Omega)$ and set

$$\boldsymbol{E}_\lambda := \chi_{|\mathbf{D}\boldsymbol{v}_\lambda| \leqslant \lambda} \boldsymbol{D}_\lambda \quad \textit{and} \quad \boldsymbol{F}_\lambda := \chi_{|\boldsymbol{D}_\lambda| > \lambda} \boldsymbol{D}_\lambda.$$

Suppose that

$$\begin{aligned} \boldsymbol{E}_\lambda &\rightharpoonup \boldsymbol{E} \quad \textit{in} \quad \boldsymbol{L}^r(\Omega), \\ \boldsymbol{S}(\boldsymbol{E}_\lambda) &\rightharpoonup \overline{\boldsymbol{S}} \quad \textit{in} \quad \boldsymbol{L}^{r'}(\Omega), \\ \lambda^\alpha \boldsymbol{F}_\lambda &\rightharpoonup \boldsymbol{F} \quad \textit{in} \quad \boldsymbol{L}^2(\Omega), \\ \lambda^{-\alpha} \boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda) &\rightharpoonup \boldsymbol{N} \quad \textit{in} \quad \boldsymbol{L}^2(\Omega). \end{aligned} \tag{2.3}$$

Then

$$
\begin{aligned}
\boldsymbol{D}_\lambda &\rightharpoonup \boldsymbol{D} \quad \textit{in} \quad \boldsymbol{L}^{\min(2,\beta,r)}(\Omega),\\
\boldsymbol{E} &= \boldsymbol{D},\\
\boldsymbol{F} &= 0,\\
\boldsymbol{N} &= 0,\\
\mathsf{S}_\lambda(\boldsymbol{D}_\lambda) &\rightharpoonup \overline{\boldsymbol{S}} \quad \textit{in} \quad \boldsymbol{L}^{\min(2,\beta',r')}(\Omega),\\
\boldsymbol{S}(\boldsymbol{E}_\lambda)\chi_{|\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda} &\rightharpoonup \overline{\boldsymbol{S}} \quad \textit{in} \quad \boldsymbol{L}^{r'}(\Omega),\\
\boldsymbol{F}_\lambda &\rightharpoonup 0 \quad \textit{in} \quad \boldsymbol{L}^{\min(2,\beta)}(\Omega),\\
\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda) &\rightharpoonup 0 \quad \textit{in} \quad \boldsymbol{L}^{\min(2,\beta')}(\Omega)
\end{aligned}
\tag{2.4}
$$

for some subsequences.

If, moreover

$$
\limsup_{\lambda\to\infty} \int_\Omega \mathsf{S}_\lambda(\boldsymbol{D}_\lambda)\cdot\boldsymbol{D}_\lambda \leqslant \int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{D}, \tag{2.5}
$$

then

$$
\overline{\boldsymbol{S}} = \boldsymbol{S}(\boldsymbol{D}).
$$

Proof. The assumption $(2.3)_3$ implies

$$
\lambda^{2\alpha}\int_{|\boldsymbol{D}_\lambda|>\lambda} |\boldsymbol{D}_\lambda|^2 \leqslant C,
$$

thus, using $\alpha > -\frac{1}{2}$, we get

$$
|\{|\boldsymbol{D}_\lambda| > \lambda\}| \lesssim \lambda^{-2\alpha-2} < \lambda^{-1} \to 0, \quad \lambda\to\infty. \tag{2.6}
$$

In this proof, for a limit identification, we shall use an arbitrary test function $\boldsymbol{B}\in\boldsymbol{L}^\infty(\Omega)$. It follows from $(2.3)_3$, (2.6) and Hölder's inequality, that

$$
\left|\int_\Omega \lambda^\alpha \boldsymbol{F}_\lambda\cdot\boldsymbol{B}\right| \lesssim |\{|\boldsymbol{D}_\lambda|>\lambda\}|^{\frac{1}{2}} \lesssim \lambda^{-\alpha-1} < \lambda^{-\frac{1}{2}} \to 0,
$$

therefore $\boldsymbol{F}=0$. Similarly, using $(2.3)_4$ and $\boldsymbol{N}_\lambda(0)=0$, we get

$$
\left|\int_\Omega \lambda^\alpha \boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)\cdot\boldsymbol{B}\right| = \left|\int_{|\boldsymbol{D}_\lambda|>\lambda} \lambda^\alpha \boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)\cdot\boldsymbol{B}\right| \lesssim \lambda^{-\alpha-1}\to 0,
$$

thus $\boldsymbol{N}=0$. To prove $(2.4)_6$, we apply $(2.3)_2$ and (2.6) to get

$$
\int_\Omega \boldsymbol{S}(\boldsymbol{E}_\lambda)\chi_{|\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}\cdot\boldsymbol{B} = \int_\Omega \boldsymbol{S}(\boldsymbol{E}_\lambda)\cdot\boldsymbol{B} - \int_{|\boldsymbol{D}_\lambda|>\lambda}\boldsymbol{S}(0)\cdot\boldsymbol{B} \to \int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{B}.
$$

With $(2.3)_3$ given, there is no need to prove $(2.4)_7$ if $\alpha\geqslant 0$. If $\alpha\in(-\frac{1}{2},0)$, then $2\alpha+2<2$ and we can use $(2.3)_3$, Hölder's inequality and (2.6) to obtain

$$
\int_\Omega |\boldsymbol{F}_\lambda|^{2\alpha+2} = \lambda^{-\alpha(2\alpha+2)}\int_\Omega |\lambda^\alpha\boldsymbol{F}_\lambda|^{2\alpha+2} \lesssim \lambda^{-\alpha(2\alpha+2)}(\lambda^{-2\alpha-2})^{-\alpha} = 1.
$$

Hence, there is a subsequence $\boldsymbol{F}_\lambda$ which converges weakly in $\boldsymbol{L}^{2\alpha+2}(\Omega)$. Its weak limit is zero, since $(2.3)_3$ gives

$$\left|\int_\Omega \boldsymbol{F}_\lambda \cdot \boldsymbol{B}\right| \lesssim \lambda^{-\alpha} \int_\Omega |\lambda^\alpha \boldsymbol{F}_\lambda| \lesssim \lambda^{-\alpha}(\lambda^{-2\alpha-2})^{\frac{1}{2}} = \lambda^{-2\alpha-1} \to 0.$$

Since

$$\int_\Omega |\boldsymbol{D}_\lambda|^p = \int_\Omega |\boldsymbol{E}_\lambda|^p + \int_\Omega |\boldsymbol{F}_\lambda|^p,$$

the properties $(2.3)_1$ and $(2.4)_7$ (that we already proved) imply $(2.4)_1$. We can prove $(2.4)_8$ similarly as $(2.4)_7$. Indeed, this property is interesting only if $\alpha > 0$ and in that case, we get

$$\int_\Omega |\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)|^{\frac{2\alpha+2}{2\alpha+1}} = \lambda^{\alpha\frac{2\alpha+2}{2\alpha+1}} \int_\Omega |\lambda^{-\alpha}\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)|^{\frac{2\alpha+2}{2\alpha+1}} \lesssim \lambda^{\alpha\frac{2\alpha+2}{2\alpha+1}}(\lambda^{-2\alpha-2})^{\frac{\alpha}{2\alpha+1}} = 1$$

and

$$\left|\int_\Omega \boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda) \cdot \boldsymbol{B}\right| \lesssim \lambda^{\alpha} \int_\Omega |\lambda^{-\alpha}\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)| \lesssim \lambda^{\alpha}(\lambda^{-2\alpha-2})^{\frac{1}{2}} = \lambda^{-1} \to 0.$$

Since

$$\int_\Omega |\mathbf{S}_\lambda(\boldsymbol{D}_\lambda)|^p = \int_{|\boldsymbol{D}_\lambda|\leqslant\lambda} |\boldsymbol{S}(\boldsymbol{E}_\lambda)|^p + \int_\Omega |\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)|^p,$$

the relation $(2.4)_5$ follows from $(2.4)_6$ and $(2.4)_8$. Using $(2.4)_1$, $(2.3)_1$ and $(2.4)_7$, we obtain

$$\int_\Omega \boldsymbol{D} \cdot \boldsymbol{B} \leftarrow \int_\Omega \boldsymbol{D}_\lambda \cdot \boldsymbol{B} = \int_\Omega \boldsymbol{E}_\lambda \cdot \boldsymbol{B} + \int_\Omega \boldsymbol{F}_\lambda \cdot \boldsymbol{B} \to \int_\Omega \boldsymbol{E} \cdot \boldsymbol{B},$$

hence $\boldsymbol{D} = \boldsymbol{E}$.

Now we shall prove the last part of the lemma, concerning the identification of $\overline{\boldsymbol{S}}$. To this end, we use the monotonicity of $\mathbf{S}_\lambda$, to write

$$\begin{aligned} 0 &\leqslant \int_\Omega (\mathbf{S}_\lambda(\boldsymbol{D}_\lambda) - \mathbf{S}_\lambda(\boldsymbol{B})) \cdot (\boldsymbol{D}_\lambda - \boldsymbol{B}) \\ &= \int_\Omega \mathbf{S}_\lambda(\boldsymbol{D}_\lambda) \cdot \boldsymbol{D}_\lambda - \int_{|\boldsymbol{D}_\lambda|\leqslant\lambda} \boldsymbol{S}(\boldsymbol{E}_\lambda) \cdot \boldsymbol{B} - \int_{|\boldsymbol{D}_\lambda|>\lambda} \boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda) \cdot \boldsymbol{B} \\ &\quad - \int_\Omega \mathbf{S}_\lambda(\boldsymbol{B}) \cdot \boldsymbol{E}_\lambda - \int_\Omega \mathbf{S}_\lambda(\boldsymbol{B}) \cdot \boldsymbol{F}_\lambda + \int_\Omega \mathbf{S}_\lambda(\boldsymbol{B}) \cdot \boldsymbol{B}. \end{aligned} \tag{2.7}$$

Now we apply limes superior $\lambda \to \infty$ to the right hand side, we use (2.5), $(2.4)_5$, $(2.4)_7$, $(2.3)_2$ and $(2.4)_6$ and get

$$0 \leqslant \int_\Omega (\overline{\boldsymbol{S}} \cdot \boldsymbol{D} - \overline{\boldsymbol{S}} \cdot \boldsymbol{B} - \boldsymbol{S}(\boldsymbol{B}) \cdot \boldsymbol{D} + \boldsymbol{S}(\boldsymbol{B}) \cdot \boldsymbol{B}) = \int_\Omega (\overline{\boldsymbol{S}} - \boldsymbol{S}(\boldsymbol{B})) \cdot (\boldsymbol{D} - \boldsymbol{B})$$

for all $\boldsymbol{B} \in \boldsymbol{L}^\infty(\Omega)$. The rest of the proof is furnished by the generalized Minty method and it can be found, e.g., in [**6**, Lemma 2.4.2]. □

The following lemma is a modification of the Du-Bois Reymond theorem and it will be used to identify the explicit boundary condition.

LEMMA 2.3. *Let $G := \bigcup_{i=1}^n G_i$, where G_i are open, pairwise disjoint sets in $\mathbb{R}^d$ of finite Lebesgue measure. Suppose that $f \in L^1(G)$ satisfies*

$$\int_G f\varphi = 0 \qquad \forall \varphi \in C_0^\infty(G), \quad \int_{G_i} \varphi = 0 \quad \forall i \in \{1, \dots, n\} \tag{2.8}$$

Then there exist $c_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, such that

$$f = c_i \qquad \text{a.e. in} \qquad G_i.$$

PROOF. Let $n = 1$ and suppose that $\psi, \eta \in C_0^\infty(G)$ with $\int_G \eta > 0$. Then the function

$$\varphi := \psi\eta - \frac{\int_G \psi\eta}{\int_G \eta}\eta$$

belongs to $C_0^\infty(G)$ and $\int_G \varphi = 0$. Consequently, using (2.8) and the properties of φ, we obtain

$$\int_G \left(f - \fint_G f \right) \psi\eta = \frac{\int_G \psi\eta}{\int_G \eta} \int_G \left(f - \fint_G f \right) \eta$$

where $\fint$ denotes the mean value of an integral. Now we are going to use this identity for a sequence of functions $0 \leqslant \eta_k \in C_0^\infty(\Theta)$, $k \in \mathbb{N}$, satisfying $\eta_k \uparrow 1$ as $k \to \infty$ pointwise in Θ. This way, if we apply the dominated convergence theorem, we get

$$\int_G \left(f - \fint_G f \right) \psi = \left(\fint_G \psi \right) \int_G \left(f - \fint_G f \right) = 0.$$

Since $\psi \in C_0^\infty(\Theta)$ was arbitrary and $f - \fint_G f \in L^1(G)$, we may infer, by means of the classical Du-Bois Reymond theorem, that $f = \fint_G f$ a.e. on G, which means that there exists a constant $c \in \mathbb{R}$ such that

$$f = c \quad \text{a.e. on } G.$$

If $n = 2$, we may choose $\varphi \in C^\infty(G)$, $\int_{G_1} \varphi = \int_{G_2} \varphi = 0$ such that it is zero in one of the components G_1, G_2 and then apply the lemma for $n = 1$. □

2.5. Model of fluid. We shall consider a stationary flow, described by the velocity field $\boldsymbol{v}$, of an incompressible (non-Newtonian) fluid in a domain Ω. The boundary $\partial\Omega$ of Ω consists of two parts: Γ, where the Dirichlet boundary condition is prescribed and Θ, which is the remaining part of the boundary. In typical situation, one may imagine a flow through a pipe, where Γ represents walls of the pipe and eventually also the inlet, if it is prescribed. On the other hand, Θ then represents the outlet, eventually also the inlet (for example if the flow is driven by a pressure difference). To save some space, we shall formulate our main results for the case, where Θ has only one component (i.e., is connected). However, we are able to easily describe also the general case of multiple outlets or inlets (see Remark 5.1).

We shall consider that, in the most general case, the stationary flow of an incompressible fluid is described by the system

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \text{in} \quad \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.9}$$

Here $\boldsymbol{S}(\mathbf{D}\boldsymbol{v})$ represents the generalized stress and $\boldsymbol{S}$ is always assumed to define a maximal monotone r-graph. If we set $\boldsymbol{T} := -p\mathbf{I} + \boldsymbol{S}(\mathbf{D}\boldsymbol{v})$ (Cauchy stress tensor), we can rewrite the momentum balance $(2.9)_2$ as

$$\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = \operatorname{div} \boldsymbol{T}.$$

3. Optimization problem

Here we shall explain the main idea. Our goal is to supplement the system (2.9) with a boundary condition on Θ. Obviously, at this moment, we need to impose some physically reasonable requirement on the outflow boundary condition we are looking for. Let us suppose that this requirement can be formulated as a minimization problem for some functional F, defined for every solution to (2.9), regardless of its boundary condition on Θ. Then the following definition has a good sense.

Definition 3.1. Let Ω and F be as above. Let $\mathcal{P}$ be a stationary system of PDE with a Dirichlet boundary condition on Γ, describing an incompressible flow in Ω. We shall say that a boundary condition on Θ for $\mathcal{P}$ is F-optimal if there exists a corresponding solution that is a minimum of F in the set of all solutions to $\mathcal{P}$.

If we, in particular, consider that F is the energy dissipation, i.e., if

$$F(\boldsymbol{v}) = \xi(\boldsymbol{v}) := \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v},$$

then a possible interpretation of Definition 3.1 is that an ξ-optimal boundary condition yields the most stable flow. It is interesting to note that the idea to minimize the energy dissipation was also used in [**7**]. However, it was used for a completely different purpose (the optimal shape of a pipe). If we take

$$F(\boldsymbol{v}) = \nu \int_\Omega |\nabla \boldsymbol{v}|^2,$$

then the physical meaning is lost. However, it has been shown in [**2**], that this choice leads to the "do-nothing" boundary condition (for the Stokes system). The last possibility which will be considered is

$$F(\boldsymbol{v}) = \eta(\boldsymbol{v}) := \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v},$$

where

$$\boldsymbol{R}(\boldsymbol{v}) := \int_0^1 \boldsymbol{S}(s\mathbf{D}\boldsymbol{v})\, \mathrm{d}s$$

which represents the potential to the energy dissipation.

If we denote by $\mathcal{S}$ the set of all weak solutions to $\mathcal{P}$, then the fact that $\boldsymbol{v}$ solves $\mathcal{P}$ with an F-optimal boundary condition can be written symbolically as

$$\boldsymbol{v} \in \arg\min_{\boldsymbol{\varphi}\in\mathcal{S}} F(\boldsymbol{\varphi}). \tag{3.1}$$

At this point, the boundary condition (3.1) is, of course, implicit.

For all this to make sense, we need to show that $\arg\min_{\boldsymbol{\varphi}\in\mathcal{S}} \xi(\boldsymbol{\varphi})$ is a non-empty set, i.e., that a F-optimal boundary condition for $\mathcal{P}$ exists. This is done in the next section for the case that $\mathcal{P}$ is the generalized Navier-Stokes, or Stokes, system.

4. The optimal outflow boundary condition in implicit form for the generalized (Navier)-Stokes system

From now on, we will consider system (2.9) or its particular cases (e.g. the Stokes case, where $\operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})$ vanishes).

Definition 4.1. Let Ω be as above. Suppose that $\boldsymbol{v}_D \in \boldsymbol{W}^{1-\frac{1}{r},r}(\partial\Omega)$ and denote by $\boldsymbol{v}_0 \in \boldsymbol{W}^{1,r}(\Omega)$ its extension. By a weak solution to (2.9), we mean a function $\boldsymbol{v} \in \boldsymbol{v}_0 + \boldsymbol{W}^{1,r}_{\Gamma,\operatorname{div}}(\Omega)$ ($\boldsymbol{v}_0 \in \boldsymbol{W}^{1,r}_{\operatorname{div}}(\Omega)$ is an extension of $\boldsymbol{v}_D$ to Ω) satisfying

$$-\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})\cdot\nabla\boldsymbol{\tau} + \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v})\cdot\mathbf{D}\boldsymbol{\tau} = 0 \quad \forall\boldsymbol{\tau}\in\boldsymbol{W}^{1,r}_{0,\operatorname{div}}(\Omega). \tag{4.1}$$

This is a meaningful definition as long as $r \geqslant \frac{3d}{d+2}$, since then $\boldsymbol{v}\otimes\boldsymbol{v}\in\boldsymbol{L}^{r'}(\Omega)$. It is, of course, possible to test (4.1) by smooth functions and then require just $r > \frac{2d}{d+2}$. However, for the sake of simplicity, we will exclude the super-critical case $r \in (\frac{2d}{d+2}, \frac{3d}{d+2})$ and from now on focus only on the sub-critical case $r \geqslant \frac{3d}{d+2}$.

The pressure p can be obtained from (4.1) in the second step by the following lemma.

Lemma 4.1. *Let $1 < r < \infty$. Suppose that $\boldsymbol{f} \in (\boldsymbol{W}^{1,r}_0(\Omega))^*$ satisfies*

$$\boldsymbol{f}(\boldsymbol{\varphi}) = 0 \quad \forall\boldsymbol{\varphi}\in\boldsymbol{W}^{1,r}_0(\Omega).$$

Then, there exists $p \in L^{r'}(\Omega)$ satisfying

$$\|p\|_{r'} \leqslant c\|\boldsymbol{f}\|_{(\boldsymbol{W}^{1,r}_0(\Omega))^*}$$

and

$$\boldsymbol{f}(\boldsymbol{\varphi}) = \int_\Omega p \operatorname{div}\boldsymbol{\varphi} \quad \forall\boldsymbol{\varphi}\in\boldsymbol{W}^{1,r}_0(\Omega).$$

Proof. For $n \in \mathbb{N}$, the expression

$$J(\boldsymbol{u}) := \frac{1}{r}\|\nabla\boldsymbol{u}\|_r^r + \frac{n}{r}\|\operatorname{div}\boldsymbol{u}\|_r^r - \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{u}\in\boldsymbol{W}^{1,r}_0(\Omega), \tag{4.2}$$

defines a coercive, weakly-$\boldsymbol{W}^{1,r}(\Omega)$ lower semi-continuous functional, which is also bounded below. Therefore, there is a minimum $\boldsymbol{u}_n \in \boldsymbol{W}_0^{1,r}(\Omega)$ of J satisfying

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t} J(\boldsymbol{u}_n + t\boldsymbol{\varphi}) \restriction_{t=0} \\
&= \int_\Omega |\nabla \boldsymbol{u}_n|^{r-2} \nabla \boldsymbol{u}_n \cdot \nabla \boldsymbol{\varphi} + n \int_\Omega |\operatorname{div} \boldsymbol{u}_n|^{r-2} \operatorname{div} \boldsymbol{u}_n \operatorname{div} \boldsymbol{\varphi} - \boldsymbol{f}(\boldsymbol{\varphi}) \qquad (4.3)
\end{aligned}
$$

for all $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,r}(\Omega)$. On testing (4.3) with $\boldsymbol{u}_n$, we obtain

$$
\begin{aligned}
&\|\nabla \boldsymbol{u}_n\|_r^r + n\|\operatorname{div} \boldsymbol{u}_n\|_r^r = \boldsymbol{f}(\boldsymbol{u}_n) \\
&\qquad \leqslant \|\boldsymbol{f}\|_{(\boldsymbol{W}_0^{1,r}(\Omega))^*} \|\boldsymbol{u}_n\|_{1,r} \leqslant c\|\boldsymbol{f}\|_{(\boldsymbol{W}_0^{1,r}(\Omega))^*} + \frac{1}{2}\|\nabla \boldsymbol{u}_n\|_r^r, \qquad (4.4)
\end{aligned}
$$

therefore

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,r}(\Omega) \qquad (4.5)$$

and

$$\operatorname{div} \boldsymbol{u}_n \to 0 \quad \text{in} \quad L^r(\Omega) \qquad (4.6)$$

for some subsequence. The relations (4.6) and (4.5) imply $\operatorname{div} \boldsymbol{u} = 0$, which, together with (4.4) and the assumptions on $\boldsymbol{f}$, gives

$$\|\nabla \boldsymbol{u}\|_r^r \leqslant \liminf_{n\to\infty} \|\nabla \boldsymbol{u}_n\|_r^r \leqslant \liminf_{n\to\infty} \boldsymbol{f}(\boldsymbol{u}_n) = \boldsymbol{f}(\boldsymbol{u}) = 0,$$

so necessarily $\boldsymbol{u} = 0$. From (4.5) we also get $\overline{\boldsymbol{S}} \in \boldsymbol{L}^{r'}(\Omega)$ and

$$|\nabla \boldsymbol{u}_n|^{r-2} \nabla \boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{S}} \quad \text{in} \quad \boldsymbol{L}^{r'}(\Omega). \qquad (4.7)$$

for another subsequence. Moreover, the relation $\nabla \boldsymbol{u}_n \rightharpoonup 0$ in $\boldsymbol{L}^r(\Omega)$ and the monotonicity of the mapping $\boldsymbol{A} \mapsto |\boldsymbol{A}|^{r-2}\boldsymbol{A}$ imply $\overline{\boldsymbol{S}} = 0$. Indeed, observe that

$$0 \leqslant \int_\Omega (|\nabla \boldsymbol{u}_n|^{r-2} \nabla \boldsymbol{u}_n - |\boldsymbol{D}|^{r-2}\boldsymbol{D}) \cdot (\nabla \boldsymbol{u}_n - \boldsymbol{D}) \to -\int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D} + \|\boldsymbol{D}\|_r^r \quad \forall \boldsymbol{D} \in \boldsymbol{L}^r(\Omega).$$

Choosing $\boldsymbol{D} := \frac{1}{2}|\overline{\boldsymbol{S}}|^{\frac{1}{r-1}-1}\overline{\boldsymbol{S}} \in \boldsymbol{L}^r(\Omega)$, we obtain

$$-\frac{1}{2}\|\overline{\boldsymbol{S}}\|_{r'}^{r'} + \frac{1}{2^r}\|\overline{\boldsymbol{S}}\|_{r'}^{r'} \geqslant 0,$$

hence $\overline{\boldsymbol{S}} = 0$.

Now let

$$p_n := n|\operatorname{div} \boldsymbol{u}_n|^{r-2} \operatorname{div} \boldsymbol{u}_n - \frac{1}{|\Omega|}\int_\Omega n|\operatorname{div} \boldsymbol{u}_n|^{r-2} \operatorname{div} \boldsymbol{u}_n \in L_0^{r'}(\Omega),$$

where $L_0^{r'}(\Omega)$ denotes the subspace of $L^{r'}(\Omega)$ whose elements g satisfy $\int_\Omega g = 0$. Since $\boldsymbol{\varphi} = 0$ on $\partial\Omega$, the divergence theorem implies that (4.3) can be rewritten as

$$\int_\Omega p_n \operatorname{div} \boldsymbol{\varphi} + \int_\Omega |\nabla \boldsymbol{u}_n|^{r-2} \nabla \boldsymbol{u}_n \cdot \nabla \boldsymbol{\varphi} = \boldsymbol{f}(\boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,r}(\Omega). \qquad (4.8)$$

Let

$$\boldsymbol{\varphi} := B\left(|p_n|^{r'-2}p_n - \frac{1}{|\Omega|}\int_\Omega |p_n|^{r'-2}p_n\right),$$

where B is Bogovskii operator. Then

$$\|\boldsymbol{\varphi}\|_{1,r} \leqslant c \left\| |p_n|^{r'-1} \right\|_r = c\, \|p_n\|_{r'}^{r'-1} \tag{4.9}$$

and

$$\int_\Omega p_n \operatorname{div} \boldsymbol{\varphi} = \|p_n\|_{r'}^{r'} - \left(\frac{1}{|\Omega|} \int_\Omega |p_n|^{r'-2} p_n \right) \left(\int_\Omega p_n \right) = \|p_n\|_{r'}^{r'}.$$

Using that, (4.8) and (4.9), we get

$$\|p_n\|_{r'}^{r'} \leqslant \|\boldsymbol{f}\|_{(\boldsymbol{W}_0^{1,r}(\Omega))^*} \|\boldsymbol{\varphi}\|_{1,r} + \|\nabla \boldsymbol{u}_n\|_r^{r-1} \|\nabla \boldsymbol{\varphi}\|_r \leqslant c \|\boldsymbol{f}\|_{(\boldsymbol{W}_0^{1,r}(\Omega))^*} \|p_n\|_{r'}^{r'-1},$$

thus

$$p_n \rightharpoonup p \quad \text{in} \quad L_0^{r'}(\Omega)$$

for some subsequence. Then, using these relations to pass to the limit in (4.8), we hereby obtain

$$\int_\Omega p \operatorname{div} \boldsymbol{\varphi} = \boldsymbol{f}(\boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,r}(\Omega)$$

and the lemma is proved. □

Hence, we may also think of a weak solution to (2.9) as of a pair $(\boldsymbol{v}, p) \in (\boldsymbol{v}_0 + \boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)) \times L^{r'}(\Omega)$ satisfying

$$-\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\tau} + \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\tau} = \int_\Omega p \operatorname{div} \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \boldsymbol{W}_0^{1,r}(\Omega). \tag{4.10}$$

In either case p is defined only up to a constant.

The functional F cannot be, of course, arbitrary. It must be coercive in the sense that

$$\lim_{\|\nabla \boldsymbol{v}\|_r \to \infty} \frac{F(\boldsymbol{v})}{\|\nabla \boldsymbol{v}\|_r} > 0. \tag{4.11}$$

Furthermore, we shall need weak lower semi-continuity, at least in the sense that

$$\begin{aligned} F(\boldsymbol{v}) \leqslant \liminf_{n\to\infty} F(\boldsymbol{v}_n) \quad & \text{for any} \quad \{\boldsymbol{v}_n\}_{n=1}^\infty \subset \mathcal{S} \\ & \text{satisfying} \quad \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \in \mathcal{S} \quad \text{in} \quad \boldsymbol{W}^{1,r}(\Omega), \end{aligned} \tag{4.12}$$

where $\mathcal{S}$ is the set of weak solutions to (2.9). If we define the topology on $\mathcal{S}$ to be the weak-$\boldsymbol{W}^{1,r}(\Omega)$ topology, then the property (4.12) reads simply as:

$$F \quad \text{is weakly lower semi-continuos on} \quad \mathcal{S}.$$

We remark that, since the theory for the existence of weak solutions for the system (2.9) with inhomogeneous Dirichlet boundary condition on $\partial\Omega$ is incomplete, the set $\mathcal{S}$ may be empty, depending on Ω, $\boldsymbol{v}_D$ and r. Therefore, in the Theorem 4.1 below we just assume that there is at least one weak solution to (4.1), without saying when this actually happens. This is not the case for the generalized Stokes system that will be considered in the next section, where the corresponding existence results are known.

THEOREM 4.1. *Let $\boldsymbol{S}$ define a maximal monotone r-graph with $r \geqslant \frac{3d}{d+2}$. If the set $\mathcal{S}$ is not empty, then the functional F satisfying* (4.11) *and* (4.12) *attains a minimum on $\mathcal{S}$.*

PROOF. Let $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ be a minimizing sequence, so that $\lim_{n\to\infty} F(\boldsymbol{v}_n) = \inf_{\mathcal{S}} F =: m$. Since we assume $\mathcal{S} \neq \emptyset$, we have $m < \infty$. Then, using a version of Korn's inequality (it is enough that the trace of $\boldsymbol{v}_n$ is prescribed on Γ), we obtain

$$\begin{aligned}\infty > F(\boldsymbol{v}_n) &> c_1(|\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n)|^r + |\mathbf{D}\boldsymbol{v}_n|^r) - c_2\\ &\geqslant c_1|\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n)|^r + c_3|\nabla(\boldsymbol{v}_n - \boldsymbol{v}_0)|^r + c_4|\mathbf{D}\boldsymbol{v}_0|^r - c_2\end{aligned}$$

for some $c, c_1, c_2 > 0$ independent of $n \in \mathbb{N}$. Hence, using the reflexivity and Lemma 4.1, we can find $\boldsymbol{v} \in \boldsymbol{v}_0 + \boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$, $\overline{\boldsymbol{S}} \in \boldsymbol{L}^{r'}(\Omega)$, $p \in L^{r'}(\Omega)$ and weakly converging subsequences satisfying

$$\begin{aligned}\boldsymbol{v}_n &\rightharpoonup \boldsymbol{v} \quad \text{in} \quad \boldsymbol{W}^{1,r}(\Omega)\\ \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n) &\rightharpoonup \overline{\boldsymbol{S}} \quad \text{in} \quad \boldsymbol{L}^{r'}(\Omega)\\ p_n &\rightharpoonup p \quad \text{in} \quad L^{r'}(\Omega).\end{aligned} \tag{4.13}$$

Moreover, the subsequence can be selected in a way that

$$\boldsymbol{v}_n \to \boldsymbol{v} \quad \text{in} \quad \boldsymbol{L}^q(\Omega) \quad \text{for} \quad q = r, 2, 3. \tag{4.14}$$

Indeed, this is a consequence of the compact embedding $\boldsymbol{W}^{1,r}(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^r(\Omega) \cap \boldsymbol{L}^3(\Omega)$ (recall that $r \geqslant \frac{3d}{d+2} > \frac{3d}{d+3}$, which implies $\frac{rd}{d-r} > 3$).

To identify the weak limit $\overline{\boldsymbol{S}}$ we shall use the maximal monotonicity of $\boldsymbol{S}$. If we use $(4.13)_2$, $(4.13)_3$ and (4.14) to pass to the limit in (4.10) (with $\boldsymbol{v}_n$ instead of $\boldsymbol{v}$), we obtain

$$-\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})\cdot\nabla\boldsymbol{\tau} + \int_\Omega \overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{\tau} = \int_\Omega p\,\mathrm{div}\,\boldsymbol{\tau} \quad \forall\boldsymbol{\tau}\in\boldsymbol{W}^{1,r}_0(\Omega). \tag{4.15}$$

Now we take $0 \leqslant \psi \in C_0^\infty(\Omega)$ and we use (4.10), $(4.13)_2$, $(4.13)_3$, (4.14) and (4.15) to compute the limit

$$\begin{aligned}&\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n)\cdot\mathbf{D}\boldsymbol{v}_n\psi\\ &= \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n)\cdot\mathbf{D}(\boldsymbol{v}_n\psi) - \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n)\cdot(\boldsymbol{v}_n\otimes\nabla\psi)\\ &= \int_\Omega (\boldsymbol{v}_n\otimes\boldsymbol{v}_n)\cdot\mathbf{D}\boldsymbol{v}_n + \int_\Omega |\boldsymbol{v}_n|^2\boldsymbol{v}_n\cdot\nabla\psi + \int_\Omega p_n\boldsymbol{v}_n\cdot\nabla\psi - \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n)\cdot(\boldsymbol{v}_n\otimes\nabla\psi)\\ &\to \int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})\cdot\mathbf{D}\boldsymbol{v} + \int_\Omega |\boldsymbol{v}|^2\boldsymbol{v}\cdot\nabla\psi + \int_\Omega p\boldsymbol{v}\cdot\nabla\psi - \int_\Omega \overline{\boldsymbol{S}}\cdot(\boldsymbol{v}\otimes\nabla\psi)\\ &= \int_\Omega \overline{\boldsymbol{S}}\cdot\mathbf{D}(\boldsymbol{v}\psi) - \int_\Omega \overline{\boldsymbol{S}}\cdot(\boldsymbol{v}\otimes\nabla\psi)\\ &= \int_\Omega \overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{v}\psi\end{aligned}$$

as $n \to \infty$. Using that, $(4.13)_1$, $(4.13)_2$ and the monotonicity of $\boldsymbol{S}$, we get

$$0 \leqslant \int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n) - \boldsymbol{S}(\boldsymbol{A})) \cdot (\mathbf{D}\boldsymbol{v}_n - \boldsymbol{A})\psi \to \int_\Omega (\overline{\boldsymbol{S}} - \boldsymbol{S}(\boldsymbol{A})) \cdot (\mathbf{D}\boldsymbol{v} - \boldsymbol{A})\psi \geqslant 0, \quad n \to \infty,$$

for all $\boldsymbol{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Since $0 \leqslant \psi \in C_0^\infty(\Omega)$ was arbitrary, we conclude that

$$(\overline{\boldsymbol{S}} - \boldsymbol{S}(\boldsymbol{A})) \cdot (\mathbf{D}\boldsymbol{v} - \boldsymbol{A}) \geqslant 0 \quad \text{a.e. in} \quad \Omega,$$

which, using the maximal monotonicity, yields $\overline{\boldsymbol{S}} = \boldsymbol{S}(\mathbf{D}\boldsymbol{v})$. The property (4.12) then gives

$$F(\boldsymbol{v}) \leqslant \liminf_{n\to\infty} F(\boldsymbol{v}_n) = \lim_{n\to\infty} F(\boldsymbol{v}_n) = m \leqslant F(\boldsymbol{v}),$$

thus $\boldsymbol{v}$ is indeed a minimum of F in $\mathcal{S}$. □

Corollary 4.1. *Let $\boldsymbol{S}$ define a maximal monotone r-graph with $r \geqslant \frac{3d}{d+2}$. If the system* (2.9) *has at least one weak solution, then there exists a weak solution to* (2.9) *with ξ-optimal boundary condition on Θ and also a weak solution to* (2.9) *with η-optimal boundary condition.*

Proof. If we prove that the dissipation ξ satisfies (4.11) and (4.12), then Theorem 4.1 finishes the proof. The property (4.11) follows immediately from the fact that $\boldsymbol{S}$ is a r-graph and from Korn's inequality. The property (4.12) follows from the monotonicity of $\boldsymbol{S}$. Indeed, if $\{\boldsymbol{v}_n\}_n$ is a sequence of weak solutions to (2.9) that converges weakly-$\boldsymbol{W}^{1,r}$ to $\boldsymbol{v}$, then we will obtain $\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n) \rightharpoonup \boldsymbol{S}(\mathbf{D}\boldsymbol{v})$ in $\boldsymbol{L}^{r'}(\Omega)$ as in the proof of Theorem 4.1. Thus, using the monotonicity of $\boldsymbol{S}$, we get

$$0 \leqslant \liminf_{n\to\infty} \int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_n) - \boldsymbol{S}(\mathbf{D}\boldsymbol{v})) \cdot (\mathbf{D}\boldsymbol{v}_n - \mathbf{D}\boldsymbol{v}) = \liminf_{n\to\infty} \xi(\boldsymbol{v}_n) - \xi(\boldsymbol{v}),$$

which proves that $\boldsymbol{v}$ is a minimum of ξ. We can proceed analogously for the functional η, except that we use the convexity of $\boldsymbol{A} \mapsto \boldsymbol{R}(\boldsymbol{A}) \cdot \boldsymbol{A}$ and property

$$\frac{\partial \boldsymbol{R}(\boldsymbol{A}) \cdot \boldsymbol{A}}{\partial \boldsymbol{A}} = \boldsymbol{S}(A).$$

□

We can also consider the generalized Stokes system, i.e.,

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega,\\ -\operatorname{div} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \text{in} \quad \Omega\\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on} \quad \Gamma. \end{aligned} \tag{4.16}$$

Then the proof of Theorem 4.1 simplifies in an obvious way and the restriction on r can be removed. Hence, we also get

Corollary 4.2. *Let $\boldsymbol{S}$ define a maximal monotone r-graph with $r > 1$. Then there exists a weak solution to* (4.16) *with a ξ-optimal boundary condition on Θ.*

Finally, note that we can restrict the set where we look for a minimum, for example, by prescribing the tangential part of the velocity on Θ. Hence, let us consider the system

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega, \\
-\operatorname{div} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \text{in} \quad \Omega \\
\boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on} \quad \Gamma \\
\boldsymbol{v}_\tau &= 0 \quad \text{on} \quad \Theta,
\end{aligned} \tag{4.17}
$$

where $\boldsymbol{v}_\tau = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ and where $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$. Since this is just an easy modification of the previous case, we eventually get

Corollary 4.3. *Let $\boldsymbol{S}$ define a maximal monotone r-graph with $r > 1$. Then there exists a weak solution to* (4.17) *with a ξ-optimal boundary condition on Θ.*

With these existence results in hand, it is natural to ask what is the explicit form of the optimal boundary condition. For the system (2.9), this question seems difficult and it is not even clear what is the source of these difficulties (besides non-linearity). So far, the most general model, for which we have been able to characterize the optimal boundary condition is the power-law model (and thus also the Stokes model as a particular case).

5. The explicit form of the optimal outflow boundary condition for the r-Stokes model

Existence results for the generalized Stokes model. First of all, we remind the reader the following lemma that is a simplified version of [**1**, Lemma 3.3]. It will play an important role in determining the explicit boundary condition in the next section. For the generalized Stokes system, it tells us which Dirichlet boundary data are reasonable to prescribe. Alternatively, it can be seen as an inverse trace theorem for solenoidal functions.

Lemma 5.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$ and $r \in (1,\infty)$. Suppose that $\boldsymbol{v}_D \in \boldsymbol{W}^{1-\frac{1}{r},r}(\partial\Omega)$ satisfies*

$$\int_{\partial\Omega} \boldsymbol{v}_D \cdot \boldsymbol{n} = 0.$$

Then, there exists $\boldsymbol{v} \in \boldsymbol{W}^{1,r}(\Omega)$, determined up to a function $\boldsymbol{u} \in \boldsymbol{W}^{1,r}_{0,\operatorname{div}}(\Omega)$, satisfying

$$\boldsymbol{v} = \boldsymbol{v}_D \quad \text{on} \quad \partial\Omega \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in} \quad \Omega.$$

Moreover, there exists a constant $C > 0$ (independent of $\boldsymbol{v}$ and $\boldsymbol{v}_D$), such that

$$\inf_{\boldsymbol{u}\in \boldsymbol{W}^{1,r}_{0,\operatorname{div}}(\Omega)} \|\boldsymbol{v}+\boldsymbol{u}\|_{1,r} \leqslant C\|\boldsymbol{v}_D\|_{1-\frac{1}{r},r}.$$

Proof. The lemma follows by a combination of [**9**, Theorem 6.9.2] with the Bogovskii operator, or with a version of the De-Rahm theorem. For details, see [**1**, Lemma 3.3] and references there. □

We also recall the following theorem that provides an existence and uniqueness of a weak solution to the generalized Stokes system with inhomogeneous Dirichlet boundary data.

THEOREM 5.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Suppose that $\boldsymbol{S}$ defines a maximally strictly monotone r-graph with $r \in (1,\infty)$. Let $\boldsymbol{v}_D \in \boldsymbol{W}^{1-\frac{1}{r},r}(\partial\Omega)$ be such that*

$$\int_{\partial\Omega} \boldsymbol{v}_D \cdot \boldsymbol{n} = 0.$$

Then, there exists an unique weak solution $(\boldsymbol{v}, p)$ (in the sense of Definition 4.1) to the system

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \quad \textit{in} \quad \Omega, \\ -\operatorname{div} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \textit{in} \quad \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \textit{on} \quad \partial\Omega. \end{aligned} \tag{5.1}$$

PROOF. By Lemma 5.1, there is $\boldsymbol{v}_0 \in \boldsymbol{W}^{1,r}_{\mathrm{div}}(\Omega)$ satisfying $\boldsymbol{v}_0 = \boldsymbol{v}_D$ on $\partial\Omega$. For almost every $\boldsymbol{x} \in \Omega$, we define

$$\boldsymbol{R}(\boldsymbol{A}) \equiv \boldsymbol{R}(\boldsymbol{x}, \boldsymbol{A}) := \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_0(\boldsymbol{x}) + \boldsymbol{A}).$$

Then $\boldsymbol{R}$ is clearly maximally strictly monotone for a.e. $\boldsymbol{x} \in \Omega$. Furthermore, from the r-coercivity of $\boldsymbol{S}$, we get

$$\begin{aligned} \boldsymbol{R}(\boldsymbol{A}) \cdot \boldsymbol{A} &= \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_0 + \boldsymbol{A}) \cdot (\mathbf{D}\boldsymbol{v}_0 + \boldsymbol{A}) - \boldsymbol{R}(\boldsymbol{A}) \cdot \mathbf{D}\boldsymbol{v}_0 \\ &\geqslant c_1(|\mathbf{D}\boldsymbol{v}_0 + \boldsymbol{A}|^r + |\boldsymbol{R}(\boldsymbol{A})|^{r'}) - c_2 - c_3|\mathbf{D}\boldsymbol{v}_0|^r - \frac{c_1}{2}|\boldsymbol{R}(\boldsymbol{A})|^{r'} \\ &\geqslant \frac{c_1}{2}(|\boldsymbol{A}|^r + |\boldsymbol{R}(\boldsymbol{A})|^{r'}) - c_4 \end{aligned}$$

for some $c_1, c_2, c_3, c_4 > 0$, hence $\boldsymbol{R}$ is also r-coercive. Thus, we may apply [**5**, Theorem 1.3.2] to get $\boldsymbol{u} \in \boldsymbol{W}^{1,r}(\Omega)$ and $p \in \boldsymbol{L}^{r'}(\Omega)$ that solve the homogeneous problem

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \\ -\operatorname{div} \boldsymbol{R}(\mathbf{D}\boldsymbol{u}) &= -\nabla p \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Now it is enough to observe that $\boldsymbol{v} := \boldsymbol{u} + \boldsymbol{v}_0$ and p is a solution to (5.1).

If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are weak solutions to (5.1), then $\boldsymbol{v}_1 - \boldsymbol{v}_2 \in \boldsymbol{W}^{1,r}_{0,\mathrm{div}}(\Omega)$, hence

$$0 = \int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_1) - \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_2)) \cdot (\mathbf{D}\boldsymbol{v}_1 - \mathbf{D}\boldsymbol{v}_2).$$

This, using the strict monotonicity of $\boldsymbol{S}$, implies $\mathbf{D}\boldsymbol{v}_1 = \mathbf{D}\boldsymbol{v}_2$ a.e. in Ω and, consequently, $\boldsymbol{v}_1 = \boldsymbol{v}_2$. □

Now we formulate one of the main results which gives us the explicit form of the η-optimal boundary condition for the generalized Stokes system.

THEOREM 5.2. *Let $\boldsymbol{S}$ define a continuous and monotone 2-graph. Furthermore, suppose that there exists its derivative $\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{A}}$ which is a symmetric tensor of fourth order satisfying*

$$\left|\frac{\partial \boldsymbol{S}(\boldsymbol{A})}{\partial \boldsymbol{A}}\right| \leqslant c_1 \tag{5.2}$$

and

$$\frac{\partial \boldsymbol{S}(\boldsymbol{A})}{\partial \boldsymbol{A}}\boldsymbol{B}\cdot\boldsymbol{B} \geqslant c_2|\boldsymbol{B}|^2 \tag{5.3}$$

with some $c_1, c_2 > 0$. Let $\boldsymbol{v}$ be a solution to

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= 0 \quad \text{in} \quad \Omega, \\ -\operatorname{div}\boldsymbol{S}(\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \text{in} \quad \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on} \quad \Gamma \end{aligned} \tag{5.4}$$

with an η-optimal boundary condition on Θ. Then, there exists a constant $c \in \mathbb{R}$, such that

$$\boldsymbol{T}\boldsymbol{n} = c\boldsymbol{n} \quad \text{on} \quad \Theta, \tag{5.5}$$

where

$$\boldsymbol{T} := -p\mathbf{I} + \boldsymbol{S}(\mathbf{D}\boldsymbol{v}).$$

PROOF. In order to utilize the optimal boundary condition, we need to construct a perturbation of $\boldsymbol{v}$ within the set of solutions. To this end, let $\varepsilon \neq 0$ and let $\boldsymbol{b} \in \boldsymbol{W}^{\frac{1}{2},2}_{\Gamma,\mathrm{div}}(\partial\Omega)$. By Lemma 5.1, there exists $\boldsymbol{w}_{\boldsymbol{b}} \in \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$, such that $\boldsymbol{w}_{\boldsymbol{b}} = \boldsymbol{b}$ on Θ. Then, we apply Theorem 5.1 to obtain a solution $(\boldsymbol{v}^\varepsilon, p^\varepsilon) \in (\boldsymbol{v}_0 + \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)) \times \boldsymbol{L}^2(\Omega)$ to the system

$$\begin{aligned} \operatorname{div}\boldsymbol{v}^\varepsilon &= 0 \quad \text{in} \quad \Omega \\ -\operatorname{div}\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}^\varepsilon) &= -\nabla p^\varepsilon \quad \text{in} \quad \Omega \\ \boldsymbol{v}^\varepsilon &= \boldsymbol{v}_0 \quad \text{on} \quad \Gamma \\ \boldsymbol{v}^\varepsilon &= \boldsymbol{v} + \varepsilon\boldsymbol{b} \quad \text{on} \quad \Theta. \end{aligned}$$

Then, we define

$$\boldsymbol{\varphi}^\varepsilon := \frac{\boldsymbol{v}^\varepsilon - \boldsymbol{v}}{\varepsilon} \in \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega).$$

If we test the equations for $\boldsymbol{v}^\varepsilon$ and $\boldsymbol{v}$ by $\boldsymbol{\varphi}^\varepsilon - \boldsymbol{w}_{\boldsymbol{b}} \in \boldsymbol{W}^{1,2}_0(\Omega)$, we get

$$\int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{S}(\mathbf{D}\boldsymbol{v}))\cdot\mathbf{D}\boldsymbol{\varphi}^\varepsilon = \int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{S}(\mathbf{D}\boldsymbol{v}))\cdot\mathbf{D}\boldsymbol{w}_{\boldsymbol{b}}.$$

Then, using the mean value theorem and dividing by ε, we obtain

$$\int_\Omega\int_0^1 \frac{\partial\boldsymbol{S}(\mathbf{D}\boldsymbol{v} + s\varepsilon\mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial\boldsymbol{A}}\mathbf{D}\boldsymbol{\varphi}^\varepsilon\cdot\mathbf{D}\boldsymbol{\varphi}^\varepsilon\,\mathrm{d}s = \int_\Omega\int_0^1 \frac{\partial\boldsymbol{S}(\mathbf{D}\boldsymbol{v} + s\varepsilon\mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial\boldsymbol{A}}\mathbf{D}\boldsymbol{\varphi}^\varepsilon\cdot\mathbf{D}\boldsymbol{w}_{\boldsymbol{b}}\,\mathrm{d}s.$$

This, together with the assumptions (5.2), (5.3) and Young's inequality, implies

$$\int_\Omega |\mathbf{D}\boldsymbol{\varphi}^\varepsilon|^2 \leqslant C, \tag{5.6}$$

hence

$$\begin{aligned} \boldsymbol{\varphi}^\varepsilon &\rightharpoonup \boldsymbol{\varphi} \quad \text{in} \quad \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega) \\ \varepsilon\boldsymbol{\varphi}^\varepsilon &\to 0 \quad \text{in} \quad \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega) \\ \boldsymbol{v}^\varepsilon &\to \boldsymbol{v} \quad \text{in} \quad \boldsymbol{W}^{1,2}_{\mathrm{div}}(\Omega) \end{aligned} \tag{5.7}$$

for some subsequences. Note that, since $\boldsymbol{A} \mapsto \boldsymbol{R}(\boldsymbol{A}) \cdot \boldsymbol{A}$ is a potential to $\boldsymbol{S}(\boldsymbol{A})$, it holds that

$$\frac{\partial \boldsymbol{R}(\boldsymbol{A})}{\partial \boldsymbol{A}} \boldsymbol{A} = \boldsymbol{S}(\boldsymbol{A}) - \boldsymbol{R}(\boldsymbol{A}).$$

Now we take $\varepsilon > 0$ and use this identity together with the assumption that $\boldsymbol{v}$ satisfies η-optimal boundary condition on Θ:

$$\begin{aligned} 0 \leqslant \frac{\eta(\boldsymbol{v}^\varepsilon) - \eta(\boldsymbol{v})}{\varepsilon} &= \frac{1}{\varepsilon} \int_\Omega (\boldsymbol{R}(\mathbf{D}\boldsymbol{v}^\varepsilon) - \boldsymbol{R}(\mathbf{D}\boldsymbol{v})) \cdot \mathbf{D}\boldsymbol{v} + \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &= \int_\Omega \int_0^1 \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{v} \, \mathrm{d}s + \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &= \int_\Omega \int_0^1 \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} (\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, \mathrm{d}s \\ &\quad - \varepsilon \int_\Omega \int_0^1 \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, s \, \mathrm{d}s + \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &= \int_\Omega \int_0^1 \boldsymbol{S}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, \mathrm{d}s - \int_\Omega \int_0^1 \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, \mathrm{d}s \\ &\quad - \varepsilon \int_\Omega \int_0^1 \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, s \, \mathrm{d}s + \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &= \varepsilon \int_\Omega \int_0^1 \int_0^1 \frac{\partial \boldsymbol{S}(\mathbf{D}\boldsymbol{v} + ts\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, \mathrm{d}t \, s \, \mathrm{d}s + \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &\quad - \varepsilon \int_\Omega \int_0^1 \int_0^1 \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + ts\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, \mathrm{d}t \, s \, \mathrm{d}s - \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &\quad - \varepsilon \int_\Omega \int_0^1 \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, s \, \mathrm{d}s + \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}^\varepsilon) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \\ &= \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon + \varepsilon \int_0^1 \int_0^1 \boldsymbol{M} \, \mathbf{D}\boldsymbol{\varphi}^\varepsilon \cdot \mathbf{D}\boldsymbol{\varphi}^\varepsilon \, \mathrm{d}t \, \mathrm{d}s, \end{aligned} \tag{5.8}$$

where

$$\boldsymbol{M} = s \frac{\partial \boldsymbol{S}(\mathbf{D}\boldsymbol{v} + ts\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} - s \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + ts\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}} + (1-s) \frac{\partial \boldsymbol{R}(\mathbf{D}\boldsymbol{v} + s\varepsilon \mathbf{D}\boldsymbol{\varphi}^\varepsilon)}{\partial \boldsymbol{A}}.$$

Using

$$\left| \frac{\partial \boldsymbol{R}(\boldsymbol{A})}{\partial \boldsymbol{A}} \right| = \left| \frac{\partial}{\partial \boldsymbol{A}} \int_0^1 \boldsymbol{S}(s\boldsymbol{A}) \, \mathrm{d}s \right| \leqslant \int_0^1 \left| \frac{\partial \boldsymbol{S}(s\boldsymbol{A})}{\partial \boldsymbol{A}} \right| \mathrm{d}s$$

and (5.2), we find $|\boldsymbol{M}| \leqslant C$. This, (5.6) and $(5.7)_1$ used in (5.8) yields

$$0 \leqslant \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\varphi}.$$

However, if we assume $\varepsilon < 0$ instead (before (5.8)), we will analogously get

$$0 \geqslant \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\varphi},$$

hence

$$0 = \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{\varphi}.$$

This implies, using the fact that $\boldsymbol{v}$ is a weak solution to (4.17), that

$$0 = \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{w}_{\boldsymbol{b}} = \int_\Omega \boldsymbol{T} \cdot \mathbf{D}\boldsymbol{w}_{\boldsymbol{b}},$$

where

$$\boldsymbol{T} = -p\mathbf{I} + \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \in \boldsymbol{L}^{r'}(\Omega).$$

This proves that the functional $T \in (\boldsymbol{W}^{\frac{1}{2},2}(\partial\Omega))^*$ defined by

$$T(\boldsymbol{b}) := \int_\Omega \boldsymbol{T} \cdot \mathbf{D}\boldsymbol{w}_{\boldsymbol{b}}$$

vanishes on the set where $\int_\Theta \boldsymbol{b} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{b} = 0$ on Γ. Therefore, the restriction of the functional T to this set can be represented by a function $g \in \boldsymbol{L}^2(\Theta)$. Since $\operatorname{div} \boldsymbol{T} = 0$, we identify $g = \boldsymbol{T}\boldsymbol{n}$. Thus, we obtain

$$\int_\Theta \boldsymbol{T}\boldsymbol{n} \cdot \boldsymbol{b} = 0 \quad \forall \boldsymbol{b} \in \boldsymbol{L}^2(\Theta), \quad \int_\Theta \boldsymbol{b} \cdot \boldsymbol{n} = 0. \tag{5.9}$$

In (5.9), if we consider only those $\boldsymbol{b}$ with $\boldsymbol{b} \cdot \boldsymbol{n} = 0$ on Θ, we get, by the Du-Bois theorem, that

$$(\boldsymbol{T}\boldsymbol{n})_\tau = 0 \quad \text{on} \quad \Theta. \tag{5.10}$$

Using that in (5.9) yields

$$\int_\Theta \boldsymbol{T}\boldsymbol{n} \cdot \boldsymbol{n}\, \phi = 0 \quad \forall \phi \in \boldsymbol{L}^2(\Theta), \quad \int_\Theta \phi = 0.$$

Lemma 2.3 then implies that there is a constant $c \in \mathbb{R}$ such that

$$\boldsymbol{T}\boldsymbol{n} \cdot \boldsymbol{n} = c \quad \text{on} \quad \Theta,$$

which together with (5.10) finishes the proof. □

Corollary 5.1. *Suppose that $\boldsymbol{S}$ defines a continuous and monotone 2-graph. Further, assume that $\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{A}}$ exists, is symmetric and satisfies (5.2) and (5.3). Then,* *there exists an unique weak solution to*

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega\\
-2\nu \operatorname{div} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \text{in} \quad \Omega\\
\boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on} \quad \Gamma\\
\boldsymbol{T}\boldsymbol{n} &= c\boldsymbol{n} \quad \text{on} \quad \Theta.
\end{aligned}
\tag{5.11}
$$

Moreover, the boundary condition on Θ *is* η*-optimal.*

Proof. By Lemma 5.1, we know that there exists a weak solution to $(5.11)_1$, $(5.11)_2$, $(5.11)_3$ with some compatible Dirichlet boundary condition on Θ. Hence, the set where η is minimized is non-empty and, by Theorem 4.1, there is a weak solution to $(5.11)_1$, $(5.11)_2$, $(5.11)_3$ with η-optimal boundary condition. By Theorem 5.2, every such solution satisfies $\boldsymbol{T}\boldsymbol{n} = c\boldsymbol{n}$ on Θ for some constant. To show uniqueness, let $(\boldsymbol{v}_1, p_1)$ and $(\boldsymbol{v}_2, p_2)$ both be weak solutions to (5.11) and let $\boldsymbol{T}_1, \boldsymbol{T}_2$ be the corresponding Cauchy stress tensors. Then $\boldsymbol{v}_1 - \boldsymbol{v}_2 \in \boldsymbol{W}^{1,2}_{\Gamma,\operatorname{div}}(\Omega)$, therefore, integrating by parts, using $(5.11)_2$ and (5.3), we get

$$
\begin{aligned}
0 &= \int_\Theta (\boldsymbol{T}_1\boldsymbol{n} - \boldsymbol{T}_2\boldsymbol{n}) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) = \int_{\partial\Omega} (\boldsymbol{T}_1\boldsymbol{n} - \boldsymbol{T}_2\boldsymbol{n}) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2)\\
&= \int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v}_1) - \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_2)) \cdot (\mathbf{D}\boldsymbol{v}_1 - \mathbf{D}\boldsymbol{v}_2)\\
&= \int_\Omega \int_0^1 \frac{\partial \boldsymbol{S}(\mathbf{D}\boldsymbol{v}_2 + s(\mathbf{D}\boldsymbol{v}_1 - \mathbf{D}\boldsymbol{v}_2))}{\partial \boldsymbol{A}} (\mathbf{D}\boldsymbol{v}_1 - \mathbf{D}\boldsymbol{v}_2) \cdot (\mathbf{D}\boldsymbol{v}_1 - \mathbf{D}\boldsymbol{v}_2)\,\mathrm{d}s\\
&\gtrsim \int_\Omega |\mathbf{D}\boldsymbol{v}_1 - \mathbf{D}\boldsymbol{v}_2|^2,
\end{aligned}
$$

hence $\boldsymbol{v}_1 = \boldsymbol{v}_2$. □

Remark 5.1. The previous result was formulated for the case, where Θ is connected (i.e., there is only one input). If there are multiple components Θ_i of Θ, we need to prescribe flow rates through Θ_i, i.e. the numbers $\int_{\Theta_i} \boldsymbol{v} \cdot \boldsymbol{n} = Q_i$ (actually, we need to prescribe this for every Θ_i, except one, due to the incompressibility). Then, a quick inspection of the proof of Theorem 5.1 reveals that the "test functions" $\boldsymbol{w}$ must satisfy $\int_{G_i} \boldsymbol{w} \cdot \boldsymbol{n} = 0$. Thus, by Lemma 2.3, we would eventually get

$$\boldsymbol{T}\boldsymbol{n} = c_i\boldsymbol{n} \quad \text{on} \quad \Theta_i,$$

where c_i are constants corresponding to the components of Θ. This remark is, of course, valid also for the results below.

Now one may ask whether there is also an analogous result directly for the energy dissipation, not its potential. The answer is positive, if we consider the r-Stokes model described by the constitutive relation

$$\boldsymbol{T} = -p\mathbf{I} + \boldsymbol{S}(\mathbf{D}\boldsymbol{v}),$$

where

$$\boldsymbol{S}(\boldsymbol{A}) := 2\nu|\boldsymbol{A}|^{r-2}\boldsymbol{A}.$$

This model is a special instance of the generalized Stokes model, where viscosity depends on $|\mathbf{D}\boldsymbol{v}|$. Then, an easy computation reveals

$$\frac{\partial \boldsymbol{S}(\boldsymbol{A})}{\partial \boldsymbol{A}}\boldsymbol{A} = (r-1)\boldsymbol{S}(\boldsymbol{A}) \quad \text{and} \quad \boldsymbol{R}(\boldsymbol{A}) = \frac{1}{r}\boldsymbol{S}(\boldsymbol{A}).$$

We shall construct the solution to the r-Stokes system as the limit of the sequence of solutions from the previous theorem. This way, the optimal boundary condition is preserved. To make this limit, we shall need the modified convergence lemmas from the second section.

The next theorem is a crucial step to find the explicit form of ξ-optimal outflow boundary condition for the r-Stokes model

THEOREM 5.3. *Let $\boldsymbol{v}_D \in \boldsymbol{W}^{1-\frac{1}{r},r}_{\mathrm{div}}(\partial\Omega)$ with $r > 2$ and denote by $\boldsymbol{v}_0 \in \boldsymbol{W}^{1,r}_{\mathrm{div}}(\Omega)$ its divergence-free extension to Ω.*

Then, there exists an unique weak solution to the system

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \quad \textit{in} \quad \Omega \\ -2\nu \operatorname{div}(|\mathbf{D}\boldsymbol{v}|^{r-2}\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \textit{in} \quad \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \textit{on} \quad \Gamma \\ \boldsymbol{T}\boldsymbol{n} &= c\boldsymbol{n} \quad \textit{on} \quad \Theta. \end{aligned} \tag{5.12}$$

Moreover, the boundary condition $(5.12)_4$ is ξ-optimal.

PROOF. The boundary condition $\boldsymbol{T}\boldsymbol{n} = c\boldsymbol{n}$ on Θ implies

$$\int_\Theta \boldsymbol{T}\boldsymbol{n}\cdot\boldsymbol{b} = 0 \quad \forall \boldsymbol{b} \in \boldsymbol{W}^{1-\frac{1}{r},r}_{\Gamma,\mathrm{div}}(\partial\Omega),$$

which gives

$$\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v})\cdot\mathbf{D}\boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega) \tag{5.13}$$

whenever $\boldsymbol{v}$ is a weak solution to (5.12). Relation (5.13) tells us that the weak formulation of the equation $-\operatorname{div}\boldsymbol{S}(\mathbf{D}\boldsymbol{v}) = -\nabla p$ remains the same if tested by functions that do not need to vanish on Θ. This is the key observation of the following proof.

The uniqueness follows by testing (5.13) with $\boldsymbol{v}_1 - \boldsymbol{v}_2 \in \boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$ and using the strict monotonicity of $\boldsymbol{S}$ (cf. the proof of Theorem 5.1).

The existence of a weak solution could be proved directly from the formulation (5.13) (using e.g. the Galerkin approximation). However, since we additionally require that our solution satisfies an ξ-optimal boundary condition, we shall construct $\boldsymbol{v}$ as certain limit of weak solutions obtained from Theorem 5.2, which we know that satisfy η-optimal boundary condition. Then, at the last step we shall use the fact that for our particular graph $\boldsymbol{S}$, we have $\eta = \frac{1}{r}\xi$. This approach, though, requires to linearize the graph $\boldsymbol{S}$.

For $\delta, \lambda \in (0,\infty)$, let

$$\mathbf{S}_{\delta\lambda}(\boldsymbol{A}) := 2\nu(\delta + \min(\lambda, |\boldsymbol{A}|^2))^{\frac{r-2}{2}}A.$$

Then, the graph $\mathbf{S}_{\delta\lambda}$ is clearly a maximally monotone 2-graph. Moreover, its derivative satisfies

$$\frac{\partial \mathbf{S}_{\delta\lambda}(\boldsymbol{A})}{\partial \boldsymbol{A}} = \begin{cases} 2\nu(r-2)|\boldsymbol{A}|^{r-4}\boldsymbol{A}\otimes\boldsymbol{A} + 2\nu(\delta+|\boldsymbol{A}|^2)^{\frac{r-2}{2}}\mathbf{I}, & |\boldsymbol{A}|<\lambda; \\ 2\nu(\delta+\lambda^2)^{\frac{r-2}{2}}\mathbf{I}, & |\boldsymbol{A}|>\lambda \end{cases}, \tag{5.14}$$

hence, it is easy to see that (5.2) and (5.3) are satisfied (with $c_2 = 2\nu(\delta+\lambda^2)^{\frac{r-2}{2}}$ and $c_1 = (r-1)c_2$). Thus, we may apply Corollary 5.1 to find the weak solution $\boldsymbol{v}_{\delta\lambda} \in \boldsymbol{v}_0 + \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$ to (5.11). To get the δ-uniform estimates for $\boldsymbol{v}_{\delta\lambda}$, we use (5.13) and the fact that $\boldsymbol{v}_{\delta\lambda} - \boldsymbol{v}_0 \in \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$ to get

$$\int_\Omega \mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})\cdot\mathbf{D}\boldsymbol{v}_{\delta\lambda} = \int_\Omega \mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})\cdot\mathbf{D}\boldsymbol{v}_0. \tag{5.15}$$

Note that, with λ fixed, the graph $\mathbf{S}_{\delta\lambda}$ is a 2-graph. Indeed we may write

$$\mathbf{S}_{\delta\lambda}(\boldsymbol{A})\cdot\boldsymbol{A} \geqslant 2\nu\min(\lambda,|\boldsymbol{A}|)^{r-2}|\boldsymbol{A}|^2 = 2\nu\lambda^{r-2}|\boldsymbol{A}|^2$$

for $|\boldsymbol{A}|$ large and

$$|\mathbf{S}_{\delta\lambda}(\boldsymbol{A})|^2 \leqslant 4\nu^2(1+\lambda^2)^{r-2}|\boldsymbol{A}|^2.$$

This, together with (5.13) and Young's inequality gives

$$\int_\Omega |\mathbf{D}\boldsymbol{v}_{\delta\lambda}|^2 + \int_\Omega |\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})|^2 \leqslant C(\lambda),$$

therefore

$$\boldsymbol{v}_{\delta\lambda} \rightharpoonup \boldsymbol{v}_\lambda \quad \text{in} \quad \boldsymbol{W}^{1,2}(\Omega) \tag{5.16}$$

and

$$\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda}) \rightharpoonup \overline{\boldsymbol{S}} \quad \text{in} \quad \boldsymbol{L}^2(\Omega) \tag{5.17}$$

as $\delta \to 0+$ for a subsequence. Using (5.17) in (5.13), we get

$$\int_\Omega \overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{w} = 0 \quad \forall \boldsymbol{w}\in\boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega). \tag{5.18}$$

We use (5.15) and (5.18) with $\boldsymbol{v}_\lambda - \boldsymbol{v}_0 \in \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$ to obtain

$$\int_\Omega \mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})\cdot\mathbf{D}\boldsymbol{v}_{\delta\lambda} = \int_\Omega \mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})\cdot\mathbf{D}\boldsymbol{v}_0 \to \int_\Omega \overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{v}_0 = \int_\Omega \overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{v}_\lambda, \tag{5.19}$$

which, in view of Lemma 2.1 is enough to conclude

$$\overline{\boldsymbol{S}} = \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda),$$

where $\mathbf{S}_\lambda(\boldsymbol{A}) := 2\nu\min(\lambda,|\boldsymbol{A}|)^{r-2}A$. At this point, we know that $\boldsymbol{v}_\lambda \in \boldsymbol{v}_0 + \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$ is a weak solution to

$$\begin{aligned} \operatorname{div}\boldsymbol{v}_\lambda &= 0 \quad \text{in} \quad \Omega \\ -\operatorname{div}\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) &= -\nabla p_\lambda \quad \text{in} \quad \Omega \\ \boldsymbol{v}_\lambda &= \boldsymbol{v}_0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{5.20}$$

where $p_\lambda \in L^2(\Omega)$ is obtained from De-Rahm's theorem. Moreover, the solution $\boldsymbol{v}_\lambda$ satisfies

$$\int_\Omega \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega). \tag{5.21}$$

Now we will prove that the boundary condition on Θ for $\boldsymbol{v}_\lambda$, that is encoded in (5.21), is η_λ-optimal.

Let

$$f(x) := 2\nu \min(\lambda, x)^{r-2}, \quad x \geqslant 0.$$

Observe that $r > 2$ and (5.14) imply

$$\frac{\partial \mathbf{S}_{\delta\lambda}(\boldsymbol{A})}{\partial \boldsymbol{A}} \boldsymbol{B} \cdot \boldsymbol{B} \geqslant f(|\boldsymbol{A}|)|\boldsymbol{B}|^2.$$

Thus, using (5.13), (5.18) tested by $\boldsymbol{v}_{\delta\lambda} - \boldsymbol{v}_\lambda \in \boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$ and (5.16), we get

$$\begin{aligned}
&\int_\Omega \int_0^1 f(|\mathbf{D}\boldsymbol{v}^s_{\delta\lambda}|)|\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda|^2 \,\mathrm{d}s \\
&\qquad \leqslant \int_\Omega \int_0^1 \frac{\partial \mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}^s_{\delta\lambda})}{\partial \boldsymbol{A}} (\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda) \cdot (\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda) \,\mathrm{d}s \\
&\qquad = \int_\Omega (\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda}) - \mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_\lambda)) \cdot (\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda) \\
&\qquad = -\int_\Omega (\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_\lambda) - \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)) \cdot (\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda) \to 0,
\end{aligned}$$

where $\mathbf{D}\boldsymbol{v}^s_{\delta\lambda} := (1-s)\mathbf{D}\boldsymbol{v}_\lambda + s\mathbf{D}\boldsymbol{v}_{\delta\lambda}$ (note that $\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_\lambda) \to \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)$ in $\boldsymbol{L}^{r'}(\Omega)$ by the dominated convergence theorem). Thus, there is a subsequence of $\{\mathbf{D}\boldsymbol{v}_{\delta\lambda}\}_\delta$, such that

$$f(|\mathbf{D}\boldsymbol{v}^s_{\delta\lambda}|)|\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda|^2 \to 0 \quad \text{a.e. in} \quad \Omega \times (0,1). \tag{5.22}$$

If, at some point where (5.22) holds, $|\mathbf{D}\boldsymbol{v}_{\delta\lambda} - \mathbf{D}\boldsymbol{v}_\lambda|$ does not converge to zero, then $f(|\mathbf{D}\boldsymbol{v}^s_{\delta\lambda}|) \to 0$ for a subsequence, which in turn implies $\mathbf{D}\boldsymbol{v}_{\delta\lambda} \to \mathbf{D}\boldsymbol{v}_\lambda = 0$. Thus, we see that in any case, we have

$$\mathbf{D}\boldsymbol{v}_{\delta\lambda} \to \mathbf{D}\boldsymbol{v}_\lambda \quad \text{a.e. in} \quad \Omega \tag{5.23}$$

for some subsequence. This, together with (5.16) and Vitali's lemma yields

$$\mathbf{D}\boldsymbol{v}_{\delta\lambda} \to \mathbf{D}\boldsymbol{v}_\lambda \quad \text{in} \quad \boldsymbol{L}^2(\Omega). \tag{5.24}$$

Using that and $\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda}) \rightharpoonup \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)$ in $\boldsymbol{L}^2(\Omega)$, we easily obtain

$$\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda}) \cdot \mathbf{D}\boldsymbol{v}_{\delta\lambda} \rightharpoonup \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda \quad \text{in} \quad \boldsymbol{L}^1(\Omega), \tag{5.25}$$

which, in turn, implies the uniform integrability of $\{\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda}) \cdot \mathbf{D}\boldsymbol{v}_{\delta\lambda}\}_\delta$. We claim that this yields also the uniform integrability of $\{\mathbf{R}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda}) \cdot \mathbf{D}\boldsymbol{v}_{\delta\lambda}\}_\delta$. Indeed, this follows immediately from the estimate

$$\begin{aligned}|\mathbf{R}_{\delta\lambda}(\boldsymbol{A})\cdot A| &\leqslant (\mathbf{R}_{\delta\lambda}(\boldsymbol{A})-\mathbf{R}_{\delta\lambda}(0))\cdot A+|\mathbf{R}_{\delta\lambda}(0)\cdot A|\\ &= \mathbf{R}_{\delta\lambda}(\boldsymbol{A})\cdot A=\int_0^1 \mathbf{S}_{\delta\lambda}(sA)\cdot(sA)\frac{1}{s}\,\mathrm{d}s\\ &\leqslant \mathbf{S}_{\delta\lambda}(\boldsymbol{A})\cdot A\int_0^1\frac{s^2}{s}\,\mathrm{d}s=\frac{1}{2}\mathbf{S}_{\lambda}(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_\lambda.\end{aligned}$$

Since $\mathbf{S}_{\delta\lambda}$ is continuous (with respect to δ, λ and its argument), its integral $\mathbf{R}_{\delta\lambda}$ must be continuous as well. Then, the property (5.23) implies $\mathbf{R}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})\cdot\mathbf{D}\boldsymbol{v}_{\delta\lambda}\to \boldsymbol{R}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_\lambda$ almost everywhere in Ω. Thus, the Vitali's theorem yields

$$\eta_{\delta\lambda}(\boldsymbol{v}_{\delta\lambda})=\int_\Omega \mathbf{R}_{\delta\lambda}(\mathbf{D}\boldsymbol{v}_{\delta\lambda})\cdot\mathbf{D}\boldsymbol{v}_{\delta\lambda}\to\int_\Omega \boldsymbol{R}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_\lambda=\eta_\lambda(\boldsymbol{v}_\lambda). \tag{5.26}$$

Now let $(\boldsymbol{u}_\lambda,\pi_\lambda)$ be a weak solution of (5.20) and let $(\boldsymbol{u}_{\delta\lambda},\pi_{\lambda\delta})$ be the weak solution to

$$\begin{aligned}-\operatorname{div}\mathbf{S}_{\delta\lambda}(\mathbf{D}\boldsymbol{u}_{\delta\lambda})&=-\nabla\pi_{\delta\lambda}\quad\text{in}\quad\Omega\\ \operatorname{div}\boldsymbol{u}_{\delta\lambda}&=0\quad\text{in}\quad\Omega\\ \boldsymbol{u}_{\delta\lambda}&=\boldsymbol{u}_\lambda\quad\text{on}\quad\partial\Omega.\end{aligned}\tag{5.27}$$

We claim that the estimates and convergence results that were proved up to this point for $\boldsymbol{v}_{\delta\lambda}$ and $\boldsymbol{v}_\lambda$ hold in an analogous form also for $\boldsymbol{u}_{\delta\lambda}$ and $\boldsymbol{u}_\lambda$. Indeed, although $\boldsymbol{u}_{\delta\lambda}$ and $\boldsymbol{u}_\lambda$ do not satisfy the implicit boundary condition of the form (5.13), we can now simply use the fact that $\boldsymbol{u}_\lambda$ is given and $\boldsymbol{u}_{\delta\lambda}-\boldsymbol{u}_\lambda\in\boldsymbol{W}^{1,2}_{0,\mathrm{div}}(\Omega)$ is a test function for the weak formulation of $(5.27)_1$. This way, we get the analogous estimates for $\boldsymbol{u}_{\delta\lambda}$ and $\boldsymbol{u}_\lambda$ and, by repeating the arguments above, we eventually find that also

$$\eta_{\delta\lambda}(\boldsymbol{u}_{\delta\lambda})\to\eta_\lambda(\boldsymbol{u}_\lambda). \tag{5.28}$$

Recall that Corollary 5.1 gives us

$$\eta_{\delta\lambda}(\boldsymbol{v}_{\delta\lambda})\leqslant\eta_{\delta\lambda}(\boldsymbol{u}_{\delta\lambda}),$$

which, using (5.26) and (5.28) immediately yields

$$\eta_\lambda(\boldsymbol{v}_\lambda)\leqslant\eta_\lambda(\boldsymbol{u}_\lambda)$$

for all weak solutions $\boldsymbol{u}_\lambda$ of (5.20).

It remains to make the limit passage $\lambda\to\infty$. This is more difficult than the $\delta\to 0+$ limit since a 2-graph transforms to r-graph, however the basic scheme of the proof is similar. From (5.21) tested by $\boldsymbol{v}_\lambda-\boldsymbol{v}_0\in\boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$, we get

$$\int_\Omega \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_\lambda=\int_\Omega \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_0, \tag{5.29}$$

which implies

$$\int_{|\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}|\mathbf{D}\boldsymbol{v}_\lambda|^r+\lambda^{r-2}\int_{|\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}|\mathbf{D}\boldsymbol{v}_\lambda|^2$$
$$\leqslant\int_{|\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}|\mathbf{D}\boldsymbol{v}_\lambda|^{r-1}|\mathbf{D}\boldsymbol{v}_0|+\lambda^{r-2}\int_{|\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}|\mathbf{D}\boldsymbol{v}_\lambda||\mathbf{D}\boldsymbol{v}_0|.$$

Here we apply the Young inequality to both integrals on the right hand side (we use $\boldsymbol{v}_0\in\boldsymbol{W}^{1,r}(\Omega)$, $r>2$) and get

$$\int_{|\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}|\mathbf{D}\boldsymbol{v}_\lambda|^r+\lambda^{r-2}\int_{|\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}|\mathbf{D}\boldsymbol{v}_\lambda|^2\leqslant C. \tag{5.30}$$

Let

$$\boldsymbol{S}(\boldsymbol{A}):=2\nu|\boldsymbol{A}|^{r-2}A$$

and

$$\boldsymbol{E}_\lambda:=\chi_{\{|\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda\}}\mathbf{D}\boldsymbol{v}_\lambda\quad\text{and}\quad\boldsymbol{F}_\lambda:=\chi_{\{|\mathbf{D}\boldsymbol{v}_\lambda|>\lambda\}}\mathbf{D}\boldsymbol{v}_\lambda.$$

We shall now verify the assumptions of Lemma 2.2 for the graph $\mathbf{S}_\lambda$. First of all, the estimate (5.30) implies that $\boldsymbol{E}_\lambda$ is bounded in $\boldsymbol{L}^r(\Omega)$ and $\lambda^{\frac{r-2}{2}}\boldsymbol{F}_\lambda$ is bounded in $\boldsymbol{L}^2(\Omega)$ (hence we choose $\alpha=\frac{r-2}{2}>-\frac{1}{2}$, $\beta=r$). Furthermore,

$$|\boldsymbol{S}(\boldsymbol{E}_\lambda)|^{r'}=2\nu||\boldsymbol{E}_\lambda|^{r-2}\boldsymbol{E}_\lambda|^{r'}=2\nu|\boldsymbol{E}_\lambda|^r,$$

thus $\boldsymbol{S}(\boldsymbol{E}_\lambda)$ is bounded in $\boldsymbol{L}^{r'}(\Omega)$ as required. Since, in our case $\boldsymbol{N}_\lambda(\boldsymbol{A})=\lambda^{r-2}A$, it also holds that

$$|\lambda^{-\frac{r-2}{2}}\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)|^2=\lambda^{-r+2}\lambda^{2(r-2)}|\boldsymbol{F}_\lambda|^2=\lambda^{r-2}|\boldsymbol{F}_\lambda|^2.$$

Therefore $\lambda^{-\frac{r-2}{2}}\boldsymbol{N}_\lambda(\boldsymbol{F}_\lambda)$ is bounded in $\boldsymbol{L}^2(\Omega)$. Hence the assumptions of the first part of Lemma 2.2 are verified and with the help of the Korn inequality we get, in particular, that there exists $\boldsymbol{v}\in\boldsymbol{v}_0+\boldsymbol{W}^{1,2}_{\Gamma,\mathrm{div}}(\Omega)$ and a subsequence such that

$$\mathbf{D}\boldsymbol{v}_\lambda\rightharpoonup\mathbf{D}\boldsymbol{v}\quad\text{in}\quad\boldsymbol{L}^2(\Omega). \tag{5.31}$$

By Lemma 2.2, the limit $\mathbf{D}\boldsymbol{v}$ coincides with the limit of $\{\boldsymbol{E}_\lambda\}_\lambda$, thus it must hold that $\mathbf{D}\boldsymbol{v}\in\boldsymbol{L}^r(\Omega)$ and $\boldsymbol{v}\in\boldsymbol{v}_0+\boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$. Further, from $(2.4)_5$ we obtain

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\rightharpoonup\overline{\boldsymbol{S}}\quad\text{in}\quad\boldsymbol{L}^{r'}(\Omega). \tag{5.32}$$

Hence, by passing to the limit in (5.21), we get

$$\int_\Omega\overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{w}=0\quad\forall\boldsymbol{w}\in\boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega). \tag{5.33}$$

Now we choose $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{v}_0\in\boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$ in (5.33) and use (5.29) together with (5.32) and the assumptions on $\boldsymbol{v}_0$ to get

$$\int_\Omega\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_\lambda=\int_\Omega\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_0\to\int_\Omega\overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{v}_0=\int_\Omega\overline{\boldsymbol{S}}\cdot\mathbf{D}\boldsymbol{v}. \tag{5.34}$$

Thus, the property (2.5) is verified, hence Lemma 2.2 yields $\overline{\boldsymbol{S}} = \boldsymbol{S}(\mathbf{D}\boldsymbol{v})$. If we use this in (5.33), we get

$$\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{w} = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}^{1,r}_{\Gamma,\mathrm{div}}(\Omega)$$

and consequently (cf. end of the proof of Theorem 5.2) also

$$\boldsymbol{T}\boldsymbol{n} = c\boldsymbol{n} \quad \text{on} \quad \Theta.$$

It remains to prove that this boundary condition for $\boldsymbol{v}$ is ξ-optimal. To this end, let $\boldsymbol{u}$ be a weak solution to the r-Stokes system and let $\boldsymbol{u}_\lambda$ be a weak solution to (5.20) with Dirichlet boundary condition $\boldsymbol{u}_\lambda = \boldsymbol{u}$ on $\partial\Omega$. Similarly as for the $\delta \to 0+$ limit, the sequence $\{\boldsymbol{u}_\lambda\}_\lambda$ satisfies analogous estimates as $\{\boldsymbol{v}_\lambda\}_\lambda$ does (since $\boldsymbol{u}_\lambda - \boldsymbol{u} \in \boldsymbol{W}^{1,2}_{0,\mathrm{div}}(\Omega)$). Since $\boldsymbol{v}_\lambda$ satisfies the η_λ-optimal boundary condition, we know that

$$\eta_\lambda(\boldsymbol{v}_\lambda) \leqslant \eta_\lambda(\boldsymbol{u}_\lambda). \tag{5.35}$$

The goal is to pass to the limit $\lambda \to \infty$ in this inequality. Again, we will only show that $\eta_\lambda(\boldsymbol{v}_\lambda) \to \frac{1}{r}\xi(\boldsymbol{v})$, since the limit on the right hand side of (5.35) is then completely analogous. At this point, we only know from Lemma 2.2, that

$$\int_\Omega \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda \to \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}, \tag{5.36}$$

i.e., convergence of $\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda$ in the sense of averages. In what follows, we shall use the special form of $\mathbf{S}_\lambda$ and the assumption $r > 2$ to prove that, actually, we have

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda \to \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{L}^1(\Omega)$$

for some subsequence. However, compared to the δ limit above, now we will only have $\mathbf{D}\boldsymbol{v}_\lambda \to \mathbf{D}\boldsymbol{v}$ in $\boldsymbol{L}^2(\Omega)$, which is insufficient to pass to the (weak) limit in the product $\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda$. To overcome this, we first verify that the sequences $\{\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_\lambda\}_\lambda$, $\{\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}\}_\lambda$ and $\{\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}\}_\lambda$ converge weakly to $\boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}$ and then we apply (5.36) and the monotonicity of $\mathbf{S}_\lambda$.

The first step is to prove that

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_\lambda \rightharpoonup \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{L}^1(\Omega). \tag{5.37}$$

To this end, we take $\phi \in \boldsymbol{L}^\infty(\Omega)$ and write

$$\begin{aligned}\int_\Omega \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_\lambda\phi &= \int\limits_{\substack{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda\\ |\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_\lambda\phi + \int\limits_{\substack{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda\\ |\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}} \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_\lambda\phi \\ &\quad + \int\limits_{\substack{|\mathbf{D}\boldsymbol{v}|>\lambda\\ |\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}} 2\nu\lambda^{r-2}\mathbf{D}\boldsymbol{v} \cdot \mathbf{D}\boldsymbol{v}_\lambda\phi + \int\limits_{\substack{|\mathbf{D}\boldsymbol{v}|>\lambda\\ |\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}} 2\nu\lambda^{r-2}\mathbf{D}\boldsymbol{v} \cdot \mathbf{D}\boldsymbol{v}_\lambda\phi \\ &=: I_1 + I_2 + I_3 + I_4\end{aligned}$$

By the dominated convergence theorem, we obtain

$$\boldsymbol{S}(\mathbf{D}\boldsymbol{v})\chi_{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda}\phi \to \boldsymbol{S}(\mathbf{D}\boldsymbol{v})\phi \quad \text{in} \quad \boldsymbol{L}^{r'}(\Omega).$$

From this and from $\boldsymbol{E}_\lambda \rightharpoonup \mathbf{D}\boldsymbol{v}$ in $\boldsymbol{L}^r(\Omega)$, we obtain

$$I_1 = \int_\Omega (\boldsymbol{S}(\mathbf{D}\boldsymbol{v})\chi_{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda}\phi) \cdot \boldsymbol{E}_\lambda \to \int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}\phi.$$

To estimate I_2, first note that

$$\boldsymbol{G}_\lambda := \lambda^{-\frac{r-2}{2}} \boldsymbol{S}(\mathbf{D}\boldsymbol{v})\chi_{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda}\chi_{|\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}\phi$$

is bounded in $\boldsymbol{L}^2(\Omega)$. Indeed, we have

$$\int_\Omega |\boldsymbol{G}_\lambda|^2 \lesssim \lambda^{-r+2} \int_{\substack{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda \\ |\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}} |\boldsymbol{S}(\mathbf{D}\boldsymbol{v})|^2 \lesssim \lambda^{-r+2}\lambda^{2(r-1)}\lambda^{-r} = 1$$

(recall that $|\{|\mathbf{D}\boldsymbol{v}_\lambda| > \lambda\}| \lesssim \lambda^{-r}$, cf. the proof of Lemma 2.2). Hence, using $\lambda^{\frac{r-2}{2}}\boldsymbol{F}_\lambda \rightharpoonup 0$ in $\boldsymbol{L}^2(\Omega)$ (again from Lemma 2.2), we get

$$I_2 = \int_\Omega \boldsymbol{G}_\lambda \cdot (\lambda^{\frac{r-2}{2}}\boldsymbol{F}_\lambda) \to 0, \quad \lambda \to \infty.$$

The term I_3 can be estimated using Hölder's inequality, $\mathbf{D}\boldsymbol{v} \in \boldsymbol{L}^r(\Omega)$ and $r > 2$ as

$$I_3 \lesssim \lambda^{r-2} \int_{\substack{|\mathbf{D}\boldsymbol{v}|>\lambda \\ |\mathbf{D}\boldsymbol{v}_\lambda|\leqslant\lambda}} |\mathbf{D}\boldsymbol{v}_\lambda|^{r'} \lesssim \lambda^{r-2}\lambda^{r'}\lambda^{-r} = \lambda^{r'-2} \to 0, \quad \lambda \to \infty.$$

To estimate the last term, first note that

$$\boldsymbol{H}_\lambda := 2\nu\lambda^{\frac{r-2}{2}}\mathbf{D}\boldsymbol{v}\chi_{|\mathbf{D}\boldsymbol{v}|>\lambda}\chi_{|\mathbf{D}\boldsymbol{v}_\lambda|>\lambda}\phi$$

is bounded in $\boldsymbol{L}^2(\Omega)$. Indeed, the Hölder's inequality and $\mathbf{D}\boldsymbol{v} \in \boldsymbol{L}^r(\Omega)$, yield

$$\int_\Omega |\boldsymbol{H}_\lambda|^2 \lesssim \lambda^{r-2}\lambda^{-r(1-\frac{2}{r})} = 1.$$

Then, using $\lambda^{\frac{r-2}{2}}\boldsymbol{F}_\lambda \rightharpoonup 0$ in $\boldsymbol{L}^2(\Omega)$ once more, we obtain

$$I_4 = \int_\Omega \boldsymbol{H}_\lambda \cdot (\lambda^{\frac{r-2}{2}}\boldsymbol{F}_\lambda) \to 0, \quad \lambda \to \infty.$$

Thus, the the property (5.37) is verified.

Next, observe that

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} \to \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{L}^1(\Omega). \tag{5.38}$$

follows by the dominated convergence theorem. Indeed, the pointwise limit is obvious and the members of the sequence are dominated by $2\nu|\mathbf{D}\boldsymbol{v}|^r \in \boldsymbol{L}^1(\Omega)$, since

$$|\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}| \leqslant |\boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}\chi_{|\mathbf{D}\boldsymbol{v}|\leqslant\lambda}| + |2\nu\lambda^{r-2}\mathbf{D}\boldsymbol{v} \cdot \mathbf{D}\boldsymbol{v}\chi_{|\mathbf{D}\boldsymbol{v}|>\lambda}| < 2\nu|\mathbf{D}\boldsymbol{v}|^r.$$

Finally, since $r > 2$, we have

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \rightharpoonup \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \quad \text{in} \quad \boldsymbol{L}^{r'}(\Omega),$$

which, using $\mathbf{D}\boldsymbol{v} \in \boldsymbol{L}^r(\Omega)$, yields

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v} \rightharpoonup \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{L}^1(\Omega). \tag{5.39}$$

By putting together (5.36), (5.37), (5.38), (5.39) and the monotonicity of $\mathbf{S}_\lambda$, we get

$$(\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) - \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v})) \cdot (\mathbf{D}\boldsymbol{v}_\lambda - \mathbf{D}\boldsymbol{v}) \to 0 \quad \text{in} \quad \boldsymbol{L}^1(\Omega), \tag{5.40}$$

and, consequently, also

$$(\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) - \mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v})) \cdot (\mathbf{D}\boldsymbol{v}_\lambda - \mathbf{D}\boldsymbol{v}) \rightharpoonup 0 \quad \text{in} \quad \boldsymbol{L}^1(\Omega). \tag{5.41}$$

This implies, using (5.37), (5.38), (5.39), that

$$\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda \rightharpoonup \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} \quad \text{in} \quad \boldsymbol{L}^1(\Omega),$$

which, in turn, means that the sequence $\{\mathbf{S}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda\}_\lambda$ is uniformly integrable.

From this point onwards, we can proceed analogously as for the δ limit. This way, we deduce that $\{\boldsymbol{R}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda)\cdot\mathbf{D}\boldsymbol{v}_\lambda\}_\lambda$ is also uniformly integrable, that $\mathbf{D}\boldsymbol{v}_\lambda \to \mathbf{D}\boldsymbol{v}$ a.e. in Ω and thus, by the Vitali's theorem, we get

$$\eta_\lambda(\boldsymbol{v}_\lambda) = \int_\Omega \boldsymbol{R}_\lambda(\mathbf{D}\boldsymbol{v}_\lambda) \cdot \mathbf{D}\boldsymbol{v}_\lambda \to \int_\Omega \boldsymbol{R}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} = \frac{1}{r}\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v} = \frac{1}{r}\xi(\boldsymbol{v}).$$

As we explained above, the corresponding property for $\boldsymbol{u}_\lambda$ can be proved analogously. Using that in (5.35), we obtain

$$\xi(\boldsymbol{v}) \leqslant \xi(\boldsymbol{u}),$$

which finishes the proof. □

Our final result gives an explicit form of ξ-optimal outflow boundary condition for the r-Stokes model.

THEOREM 5.4. *Let $r \geqslant 2$ and let $(\boldsymbol{v}, p)$ be a weak solution to*

$$\begin{aligned} -2\nu \operatorname{div}(|\mathbf{D}\boldsymbol{v}|^{r-2}\mathbf{D}\boldsymbol{v}) &= -\nabla p \quad \text{in} \quad \Omega \\ \operatorname{div} \boldsymbol{v} &= 0 \quad \text{in} \quad \Omega \\ \boldsymbol{v} &= \boldsymbol{v}_D \quad \text{on} \quad \Gamma \end{aligned} \tag{5.42}$$

with a ξ-optimal boundary condition on Θ. Then

$$\boldsymbol{T}\boldsymbol{n} = c\boldsymbol{n} \quad \text{on} \quad \Theta. \tag{5.43}$$

PROOF. The case $r = 2$ was done in [**2**].

If we prove that there is only one solution to $(5.42)_1$, $(5.42)_2$, $(5.42)_3$ with a ξ-optimal boundary condition, then it must be the solution that was constructed in Theorem 5.3, which satisfies (5.43). Hence, it remains to prove the uniqueness.

Let $\boldsymbol{v}_1 \neq \boldsymbol{v}_2$ be weak solutions to $(5.42)_1$, $(5.42)_2$, $(5.42)_3$ with a ξ-optimal boundary condition. Then, necessarily, $\xi(\boldsymbol{v}_1) = \xi(\boldsymbol{v}_2)$. Let $\boldsymbol{v}$ be another solution to $(5.42)_1$, $(5.42)_2$, $(5.42)_3$ with the Dirichlet boundary condition $\boldsymbol{v} = \frac{\boldsymbol{v}_1+\boldsymbol{v}_2}{2}$ on $\partial\Omega$ (which exists due to Theorem 5.1). Since $\boldsymbol{S}(\boldsymbol{A}) := 2\nu|\boldsymbol{A}|^{r-2}A$ is strictly monotone, the mapping $f(\boldsymbol{A}) := \boldsymbol{S}(\boldsymbol{A}) \cdot \boldsymbol{A}$ is strictly convex as

$$\left(\frac{\partial f(\boldsymbol{A})}{\partial \boldsymbol{A}} - \frac{\partial f(\boldsymbol{B})}{\partial \boldsymbol{A}}\right) \cdot (\boldsymbol{A} - \boldsymbol{B}) = r(\boldsymbol{S}(\boldsymbol{A}) - \boldsymbol{S}(\boldsymbol{B})) \cdot (\boldsymbol{A} - \boldsymbol{B}) > 0$$

whenever $\boldsymbol{A} \neq \boldsymbol{B}$. Thus, we obtain

$$f(\boldsymbol{A}) - f(\boldsymbol{B}) > \frac{\partial f(\boldsymbol{B})}{\partial \boldsymbol{A}} \cdot (\boldsymbol{A} - \boldsymbol{B}) = r\boldsymbol{S}(\boldsymbol{B}) \cdot (\boldsymbol{A} - \boldsymbol{B})$$

and, by choosing $\boldsymbol{A} = \mathbf{D}\boldsymbol{v}_i$, $i = 1, 2$, $\boldsymbol{B} = \mathbf{D}\boldsymbol{v}$, we get

$$\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_i < \frac{1}{r}\xi(\mathbf{D}\boldsymbol{v}_i) + \frac{1}{r'}\xi(\mathbf{D}\boldsymbol{v}).$$

Using that and the fact that $\boldsymbol{v}$ is a weak solution and $\boldsymbol{v} - \frac{\boldsymbol{v}_1 + \boldsymbol{v}_2}{2}$ a test function, we find

$$\begin{aligned} \xi(\boldsymbol{v}) &= \frac{1}{2}\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_1 + \frac{1}{2}\int_\Omega \boldsymbol{S}(\mathbf{D}\boldsymbol{v}) \cdot \mathbf{D}\boldsymbol{v}_2 \\ &< \frac{1}{2r}\xi(\boldsymbol{v}_1) + \frac{1}{2r}\xi(\boldsymbol{v}_2) + \frac{1}{r'}\xi(\boldsymbol{v}) = \frac{1}{r}\xi(\boldsymbol{v}_1) + \frac{1}{r'}\xi(\boldsymbol{v}), \end{aligned}$$

hence

$$\xi(\boldsymbol{v}) < \xi(\boldsymbol{v}_1),$$

which is the desired contradiction. □

LEMMA 5.2. *Let $\boldsymbol{b} \in \boldsymbol{W}^{\frac{1}{2},2}(\partial\Omega)$ such that $\int_{\partial\Omega} \boldsymbol{b} \cdot \boldsymbol{n} = 0$. There exists a constant $\delta > 0$ (depending on d, Ω) such that if*

$$\|\boldsymbol{b}\|_{\boldsymbol{W}^{\frac{1}{2},2}_{\mathrm{div}}(\partial\Omega)} < \delta,$$

then there exists a solution $\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega)$ satisfying

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \quad \textit{in} \quad \Omega \\ \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) &= \operatorname{div} \boldsymbol{T} \quad \textit{in} \quad \Omega \\ \boldsymbol{v} &= \boldsymbol{b} \quad \textit{on} \quad \partial\Omega. \end{aligned}$$

PROOF. □

References

1. Ch. Amrouche, V. Girault: *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension.* Czechoslovak Math. J. 44(119) (1994), no. 1, 109–140.
2. M. Bathory: *Outflow Boundary Condition Leading to Minimal Energy Dissipation for an Incompressible Flow*, WDS'17 Proceedings, Prague, Matfyzpress, pp. 7–12 , 2017.
3. M. Braack, P. Mucha: *Directional do-nothing condition for the Navier-Stokes equations.* J. Comput. Math. 32 (2014), no. 5, 507–521.
4. C.-H. Bruneau, P. Fabrie: *New efficient boundary conditions for incompressible Navier-Stokes equations: a well-posedness result.* RAIRO Modél. Math. Anal. Numér. 30 (1996), no. 7, 815–840.
5. M. Bulíček, P. Gwiazda, J. Málek, K.R. Rajagopal and A. Swierczevska-Gwiazda: *On flows of fluids described by an implicit constitutive equation characterized by a maximal monotone graph*, In Mathematical Aspects of Fluid Mechanics, Cambridge University Press, 2012, 26–54.
6. M. Bulíček, P. Gwiazda, J. Málek and A. Swierczevska-Gwiazda: *On Unsteady Flows of Implicitly Constituted Incompressible Fluids*, SIAM J. Math. Anal., **44**, No. 4, 2756–2801, 2012.
7. A. Henrot, Y. Privat: *What is the optimal shape of a pipe?* Arch. Ration. Mech. Anal. 196 (2010), no. 1, 281–302.

8. Heywood, J. G., Rannacher, R., Turek, S., *Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations*, Internat. J. Numer. Methods Fluids 22 (1996), 5, 325–352.
9. A. Kufner, O. John, S. Fučík: *Function spaces.* Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977.

Department of Mathematics, Charles University, Prague, Czech Republic
Email address: `bathory@karlin.mff.cuni.cz`

Department of Mathematics, Charles University, Prague, Czech Republic
Email address: `mbul8060@karlin.mff.cuni.cz`