

Universal Algebra Exercises - Sheet 4

If you have any questions about the exercises, feel free to email me at vojtech.david@matfyz.cuni.cz or ask during the practicals.

Exercise 1. Show that the lattice $\text{Con}(\mathbf{A})$ is algebraic and give explicit descriptions of $\alpha \wedge \beta$ and $\alpha \vee \beta$ for $\alpha, \beta \in \text{Con}(\mathbf{A})$.

Exercise 2. Let $\mathbf{A} = (A, *)$ be a binary algebra and θ an equivalence relation on A . Show that θ is a congruence relation if and only if for all $a, b, c \in A$ we have

$$(a, b) \in \theta \implies \begin{cases} (a * c, b * c) \in \theta & \text{and} \\ (c * a, c * b) \in \theta. \end{cases}$$

Exercise 3. Let $\mathbf{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

$*$	0	1	2	3
0	0	2	1	1
1	2	1	0	2
2	1	0	2	0
3	1	2	0	3

Draw the lattice of subalgebras and the lattice of congruences of $\mathbf{A}$.

Exercise 4. Consider the algebra $\mathbf{A} = (\mathbb{Z}, +, \cdot) \times (\mathbb{Z}, \cdot, +)$. What is $\text{Sg}_{\mathbf{A}}((0, 1), (1, 0))$?

Exercise 5. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras in the same signature and let $f : \mathbf{A} \rightarrow \mathbf{B}$ be a homomorphism.

- Given two subuniverses $U \leq \mathbf{A}$ and $V \leq \mathbf{B}$, are $f(U) \subseteq \mathbf{B}$ and $f^{-1}(V) \subseteq \mathbf{A}$ subuniverses?
- Given two congruences $\theta \in \text{Con}(\mathbf{A})$ and $\psi \in \text{Con}(\mathbf{B})$, is $f(\theta) \in \text{Con}(\mathbf{B})$ and $f^{-1}(\psi) \in \text{Con}(\mathbf{A})$?
- Given a subset $X \subseteq A$ is $f(\text{Sg}_{\mathbf{A}}(X)) = \text{Sg}_{\mathbf{B}}(f(X))$?

Exercise 6. Given a binary algebra $\mathbf{A} = (A, *)$ define its *nucleus* as

$$B := \{a \in A \mid \forall x, y \in A, (x * a) * y = x * (a * y)\}.$$

Show that B is a subuniverse of $\mathbf{A}$. Bonus: find an example of an algebra $\mathbf{A}$ whose nucleus is empty.

Exercise 7. Recall that a Boolean algebra is a bounded distributive lattice with the greatest element 1 and the least element 0 expanded by a unary *complementation* operation $\neg$ satisfying the identities:

$$a \wedge \neg a \approx 0 \quad \text{and} \quad a \vee \neg a \approx 1.$$

Let $\mathbf{B} = (\{0, 1\}, \wedge, \vee, \neg, 0, 1)$ be the two-element Boolean algebra. Show that

$$(2^X, \cap, \cup, ^c, \emptyset, X) \simeq \mathbf{B}^X$$

for every set X . Here, $A^c = X \setminus A$ is the usual set complementation operation.